Commutative Algebra notes

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2026

Abstract

Notes taken while studying Commutative Algebra, mostly from Atiyah & MacDonald book [1] and Reid's book [2]. For the exercises, I follow the assignments listed at [3].

Usually while reading books and papers I take handwritten notes in a notebook, this document contains some of them re-written to $LaTeX$.

The proofs may slightly differ from the ones from the books, since I try to extend them for a deeper understanding.

Contents

1 Ideals

1.1 Definitions

Definition (ideal)**.** $I \subset R$ (R ring) such that $0 \in I$ and $\forall x \in I,\ r \in R,\ xr, rx \in I$.

ie. I absorbs products in R.

Definition (prime ideal)**.** if $a, b \in R$ with $ab \in P$ and $P \neq R$ (P a prime ideal), implies $a \in P$ or $b \in P$.

Definition (principal ideal)**.** generated by a single element, (a).
(a): principal ideal, the set of all multiples xa with $x \in R$.

Definition (maximal ideal)**.** $\mathfrak{m} \subset A$ (A ring) with $m \neq A$ and there is no ideal I strictly between $\mathfrak{m}$ and A. ie. if $\mathfrak{m}$ maximal and $\mathfrak{m} \subseteq I \subseteq A$, either $\mathfrak{m} = I$ or $I = A$.

Definition (unit)**.** $x \in A$ such that $xy = 1$ for some $y \in A$. ie. element *which divides 1.*

Corollary 1.8. $A = A^{\times} \sqcup \bigcup m$ (where $\sqcup$ denotes "disjoint union"), ie. $f \in A$ is either a unit or it is contained in a maximal ideal, not both.

Definition (zerodivisor)**.** $x \in A$ such that $\exists\ 0 \neq y \in A$ such that $xy = 0 \in A$. ie. *x divides 0.*.
If a ring does not have zerodivisors is an integral domain.

Definition (prime spectrum - $Spec(A)$)**.** set of prime ideals of A. ie.

$$Spec(A) = \{P \mid P \subset A \text{ is a prime ideal}\}$$

Definition (integral domain)**.** Ring in which the product of any two nonzero elements is nonzero.
ie. no zerodivisors.
ie. $\forall\ 0 \neq a,\ 0 \neq b \in A,\ ab \neq 0 \in A$.
Every field is an integral domain, not the converse.

Definition (principal ideal domain - PID)**.** integral domain in which every ideal is principal. ie. ie. $\forall I \subset R,\ \exists\ a \in I$ such that $I = (a) = \{ra \mid r \in R\}$.

Definition (nilpotent)**.** $a \in A$ such that $a^n = 0$ for some $n > 0$.

Definition (nilrad A)**.** set of all nilpotent elements of A; is an ideal of A.

if $nilradA = 0 \implies A$ has no nonzero nilpotents.

$$nilradA = \bigcap_{P \in Spec(A)} P$$

Definition (idempotent)**.** $e \in A$ such that $e^2 = e$.

Definition (radical of an ideal)**.**

$$radI = \{f \in A | f^n \in I \text{ for some } n\}$$

$radI$ is an ideal.

$nilradA = rad0$

$radI = \bigcap_{\substack{P \in \mathrm{Spec}(A) \\ P \supset I}} P$

Definition 1.13 (local ring)**.** A ring is *local* if it has a unique maximal ideal.

Notation: local ring A, its maximal ideal $\mathfrak{m}$, residue field $K = A/\mathfrak{m}$:

$$A \supset \mathfrak{m} \text{ or } (A, \mathfrak{m}) \text{ or } (A, \mathfrak{m}, K)$$

By Corollary 1.8, A is local

$\iff A$ has only one maximal ideal.

$\iff$ all the nonunits of A form an ideal.

1.2 Z and K[X], two Principal Ideal Domains

Lemma . $\mathbb{Z}$ is a PID.

Proof. Let I a nonzero ideal of $\mathbb{Z}$.

Since $I \neq \{0\}$, there is at least one nonzero integer in I. Choose the smallest element of I, namely d.

Observe that $(d) \subseteq I$, since $d \in I$. Then, every multiple $nd \in I$, since I is an ideal.

Take $a \in I$. By the Euclidean division algorithm in $\mathbb{Z}$, $a = qd + r$, with $q, r \in \mathbb{Z}$ and $0 \leq r \leq d$.

Then $r = a - qd \in I$, but d was chosen to be the smallest positive element of I, so the only possibility is $r = 0$.

Hence, $a = qd$, so $a \in (d)$, giving $I \subseteq (d)$.

Since we had $(d) \subseteq I$ and now we got $I \subseteq (d)$, we have $I = (d)$, so every ideal of $\mathbb{Z}$ is principal. Thus $\mathbb{Z}$ is a Principal Ideal Domain(PID). □

Lemma . $K[X]$ is a PID.

Proof. This proof follows very similarly to the previous proof.

Let K be a field, $K[X]$ a polynomial ring.

Take $\{0\} \neq I \subseteq K[X]$.

Since $I \neq \{0\}$, there is at least one non-zero polynomial in I.

Let $p(X) \in I$ be of minimal degree among nonzero elements of I.

Observe that $(p(X)) \subseteq I$, because $p(X) \in I$ and I is an ideal.

Let $f(X) \in I$. By Euclidean division algorithm in $K[X]$, $\exists q, r \in K[X]$ such that $f(X) = q(X) \cdot p(X) + r(X)$ with eithr $r(X) = 0$ or $deg(r) < deg(p)$.

Since $f, p \in I$, then $r(X) = f(X) - q(X) \cdot p(X) \in I$

If $r(X) \neq 0$, then $deg(r) < deg(p)$, which contradicts the minimality of $deg(p)$ in I.

Therefore, $r(X) = 0$, thus $f(X) = q(X) \cdot p(X)$, hence $f(X) \in (p(X))$. Henceforth, $I \subseteq (p(X))$.

Then, since $(p(X)) \subseteq I$ and $I \subseteq (p(X))$, we have that $I = (p(X))$.

So every ideal of $K[X]$ is principal; thus $K[X]$ is a PID.

□

1.3 Zorn's lemma and Jacobson radicals

Let Σ be a partially orddered set. Given subset $S \subset \Sigma$, an *upper bound* of S is an element $u \in \Sigma$ such that $s < u \forall s \in S$.

A *maximal element* of Σ, is $m \in \Sigma$ such that $m < s$ does not hold for any $s \in \Sigma$.

A subset $S \subset \Sigma$ is *totally ordered* if for every pair $s_1, s_2 \in S$, either $s_1 \leq s_2$ or $s_2 \leq s_1$.

Lemma R.1.7 (Zorn's lemma)**.** Suppose Σ a nonempty partially ordered set (ie. we are given a relation $x \leq y$ on Σ), and that any totally ordered subset $S \subset \Sigma$ has an upper bound in Σ.

Then Σ has a maximal element.

Theorem AM.1.3. Every ring $A \neq 0$ has at least one maximal ideal.

Proof. By Zorn's lemma R.1.7. □

Corollary AM.1.4. if $I \neq (1)$ an ideal of A, $\exists$ a maximal ideal of A containing I.

Corollary AM.1.5. Every non-unit of A is contained in a maximal ideal.

Definition (Jacobson radical)**.** The *Jacobson radical* of a ring A is the intersection of all the maximal ideals of A.

Denoted $Jac(A)$.

$Jac(A)$ is an ideal of A.

Proposition AM.1.9. $x \in Jac(A)$ iff $(1 - xy)$ is a unit in A, $\forall y \in A$.

Proof. Suppose $1 - xy$ not a unit.

By AM.1.5, $1 - xy \in \mathfrak{m}$ for $\mathfrak{m}$ some maximal ideal.

But $x \in Jac(A) \subseteq \mathfrak{m}$, since $Jac(A)$ is the intersection of all maximal ideals of A.

Hence $xy \in \mathfrak{m}$, and therefore $1 \in \mathfrak{m}$, which is absurd, thus $1 - xy$ is a unit.

Conversely:
Suppose $x \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

Then $\mathfrak{m}$ and x generate the unit ideal (1), so that we have $u + xy = 1$ for some $u \in \mathfrak{m}$ and some $y \in A$.

Hence $1 - xy \in \mathfrak{m}$, and is therefore not a unit. □

2 Modules

2.1 Modules concepts

Let A be a ring. An A-module is an Abelian group M with a multiplication map

$$\begin{aligned} A \times M &\longrightarrow M \\ (f, m) &\longmapsto fm \end{aligned}$$

satisfying $\forall\ f, g \in A, \quad m, n \in M$.

i. $f(m \pm n) = fm \pm fn$

ii. $(f \pm g)m = fm \pm gm$

iii. $(fg)m = f(gm)$

iv. $1_A m = m$

Let $\psi : M \longrightarrow M$ an A-linear endomorphism of M.
$A[\psi] \subset EndM$ is the subring geneerated by A and the action of ψ.

- since ψ is A-linear, $A[\psi]$ is a commutative ring.
- M is a module over $A[\psi]$, so ψ beomes multiplication by a ring element.

2.2 Cayley-Hamilton theorem, Nakayama lemma, and corollaries

Proposition AM.2.4. (Cayley-Hamilton Theorem) Let M a fingen A-module. Let $\mathfrak{a}$ an ideal of A, let ψ an A-module endomorphism of M such that $\psi(M) \subseteq \mathfrak{a}M$.

Then ψ satisfies

$$\psi^n + a_1\psi^{n-1} + \ldots + a_{n-1}\psi + a_n = 0$$

with $a_i \in \mathfrak{a}$.

Proof. Since M fingen, let $\{x_1, \ldots, x_n\}$ be generators of M.
By hypothesis, $\psi(M) \subseteq \mathfrak{a}M$; so for any generator x_i, it's image $\psi(x_i) \in \mathfrak{a}M$.

Any element in $\mathfrak{a}M$ is a linear combination of the generators with coefficients in the ideal $\mathfrak{a}$, thus

$$\psi(x_i) = \sum_{j=1}^{n} a_{ij} x_j$$

with $a_{ij} \in \mathfrak{a}$.

Thus, for a module with n generators, we have n different $\psi(x_i)$ equations:

$$\left.\begin{aligned} \psi(x_1) &= a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \\ \psi(x_2) &= a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \\ &\ldots \\ \psi(x_n) &= a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n \end{aligned}\right\} \begin{array}{l}\text{n elements } \psi(x_i) \in \mathfrak{a}M \text{ which} \\ \text{are linear combinations of the} \\ n \text{ generators of } M\end{array}$$

Next step: rearrange in order to use matrix algebra.

Observe that each row equals 0, and rearranging the elements at each row we get

$$\left.\begin{aligned} &\psi(x_1) - (a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n) = 0 \\ &\psi(x_2) - (a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n) = 0 \\ &\ldots \\ &\psi(x_n) - (a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n) = 0 \end{aligned}\right\}$$

Then, group the x_i terms together; as example, take the row $i = 1$:

$$(\psi - a_{1,1})x_1 - a_{1,2}x_2 - \ldots - a_{1,n}x_n = 0$$

$$\left.\begin{aligned} (\psi - a_{1,1})x_1 - a_{1,2}x_2 - \ldots - a_{1,n}x_n &= 0 \\ -a_{2,1}x_1 + (\psi - a_{2,2})x_2 - \ldots - a_{2,n}x_n &= 0 \\ \ldots \\ -a_{1,1}x_1 - a_{1,2}x_2 - \ldots + (\psi - a_{1,n})x_n &= 0 \end{aligned}\right\}$$

So, $\forall i \in [n]$, as a matrix:

$$\begin{pmatrix} \psi - a_{1,1} & -a_{1,2} & \ldots & -a_{1,n} \\ -a_{2,1} & \psi - a_{2,2} & \ldots & -a_{2,n} \\ \vdots & & & \\ -a_{n,1} & -a_{n,2} & \ldots & \psi - a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Denote the previous matrix by Φ. Let m denote the vector $(x_1, x_2, \ldots, x_n)^T$ (ie. the vector of generators of the A-module M).

Then we can write the previous equality as

$$\Phi \cdot m = 0 \tag{1}$$

We know that

$$adj(\Phi)\Phi = det(\Phi)I \tag{2}$$

(aka. *fundamental identity for the adjugate matrix*).

So if at (1) we multiply both sides by $adj(\Phi)$,

$$\begin{aligned} adj(\Phi)\cdot\Phi{\cdot}m &= 0 \\ \text{(recall from (2): } adj(\Phi)\Phi &= det(\Phi)\cdot I \text{)} \\ = det(\Phi)\cdot I{\cdot}m &= 0 \end{aligned}$$

Thus,

$$det(\Phi)\cdot I{\cdot}m = 0 :$$
$$\begin{pmatrix} det(\Phi) & 0 & \dots & 0 \\ 0 & det(\Phi) & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & det(\Phi) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\implies$

$$det(\Phi)\cdot x_i = 0 \;\; \forall i \in [n] \tag{3}$$

ie. $det(\Phi)$ is an *annihilator* of the generators x_i of M, thus is an annihilator of the entire module M.

So, we're interested into calculating the $det(\Phi)$.

By the Leibniz formula,

$$\det(A) = \sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$$

thus,

$$det(\Phi) = \underbrace{(\psi - a_{11})(\psi - a_{22})\dots(\psi - a_{nn})}_{\text{diagonal of }\Phi\text{, leading term of the determinant}} - \dots$$

The *determinant trick* is that the terms that go after the "leading term of the determinant", will belong to $\mathfrak{a}$ and their combinations with ψ will not be bigger than ψ^n. Furthermore, when expanding it

- highest power is $1\cdot\psi^n$
- coefficient of ψ^{n-1} is $-(\underbrace{a_{11} + a_{22} + \dots + a_{nn}}_{a_1})$,

  where, since each $a_{ii} \in \mathfrak{a}$, $\;a_1 \in \mathfrak{a}$
- the rest of coefficients of ψ^k are also elements in $\mathfrak{a}$

Therefore we have

$$det(\Phi) = \psi^n + a_1\psi^{n-1} + a_2\psi^{n-2} + \dots + a_{n-1}\psi + a_n$$

with $a_i \in \mathfrak{a}$.

Now, notice that we had $det(\Phi) \cdot x_i = 0 \; \forall \; i \in [n]$.

The matrix Φ is the *characteristic matrix*, $xI - A$, viewed as an operator. Then,

$$det(\Phi) = det(xI - A) = p(x)$$

where $p(x)$ is the *characteristic polynomial.*

If a linear transformation turns every basis vector (x_i) into zero, then that transformation is the zero transformation. So in our case, $det(\Phi)$ is the zero transformation, thus $det(\Phi) = 0$. Therefore,

$$\psi^n + a_1\psi^{n-1} + a_2\psi^{n-2} + \ldots + a_{n-1}\psi + a_n = 0$$

□

Corollary AM.2.5. Let M a fingen A-module, let $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}M = M$.

Then, $\exists \; x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.

Proof. take $\psi =$ identity. Then in Cayley-Hamilton (AM.2.4):

$$\begin{aligned} &\psi^n + a_1\psi^{n-1} + a_2\psi^{n-2} + \ldots + a_{n-1}\psi + a_n = 0 \\ \implies &id_M + a_1 id_M + a_2 id_M + \ldots + a_{n-1} id_M + a_n = 0 \\ \implies &(1 + a_1 + \ldots + a_n) id_M = 0 \end{aligned}$$

apply it to $m \in M$, where since $id_M(m) = m$ (by definition of the identity), we then have

$$(1 + a_1 + \ldots + a_n) \cdot m = 0$$

with $a_i \in \mathfrak{a}$.

part i. $xM = 0$:
Thus the scalar $x = (1 + a_1 + \ldots + a_n)$ annihilates every $m \in M$, ie. the entire module M.

part ii. $x \equiv 1 \pmod{\mathfrak{a}}$:
$x \equiv 1 \pmod{\mathfrak{a}} \iff (x - 1) \in \mathfrak{a}$
then from $x = (1 + \underbrace{a_1 + \ldots + a_n}_{b}) \in \mathfrak{a}$, set $b = a_1 + \ldots + a_n$,
so that $x = (1 + b) \in \mathfrak{a}$.

Then $x - 1 = (1 + b) - 1 = b \in \mathfrak{a}$
so $x - 1 \in \mathfrak{a}$, thus $x \equiv 1 \pmod{\mathfrak{a}}$ as stated.

□

Proposition AM.2.6 (Nakayama's lemma)**.** Let M a fingen A-module, let $\mathfrak{a}$ an ideal of A such that $\mathfrak{a} \subseteq Jac(A)$.

Then $\mathfrak{a}M = M$ implies $M = 0$.

Proof. By AM.2.5: since $\mathfrak{a}M = M$, we have $xM = 0$ for some $x \equiv 1 \pmod{Jac(A)}$. (notice that at AM.2.5 is $\pmod{\mathfrak{a}}$ but here we use $\pmod{Jac(A)}$, since we have $\mathfrak{a} \subseteq Jac(A)$).

(recall AM.1.9: $x \in Jac(A)$ iff $(1 - xy)$ is a unit in A, $\forall y \in A$).
By AM.1.9, x is a unit in A (thus $x^{-1} \cdot x = 1$).

Hence $M = x^{-1} \cdot \underbrace{x \cdot M}_{=0 \text{ (by AM.2.5)}} = 0$.

Thus, if $\mathfrak{a}M = M$ then $M = 0$. □

Corollary AM.2.7. Let M a fingen A-module, let $N \subseteq M$ a submodule of M, let $\mathfrak{a} \subseteq Jac(A)$ an ideal.

Then $M = \mathfrak{a}M + N \overset{\text{implies}}{\Longrightarrow} M = N$.

Proof. The idea is to apply Nakayama (AM.2.6) to M/N.

Since M fingen $\Longrightarrow$ M/N is fingen and an A-module.

Since $\mathfrak{a} \subseteq Jac(A)$ $\Longrightarrow$ Nakayama applies to M/N too.

By definition,

$$\mathfrak{a}M = \left\{ \sum a_i \cdot m_i \;\mid\; a_i \in \mathfrak{a}, m_i \in M \right\}$$

where m_i are the generators of M.

Then, for M/N,

$$\mathfrak{a}(\frac{M}{N}) = \left\{ \sum a_i \cdot (m_i + N) \;\mid\; a_i \in \mathfrak{a}, m_i \in M \right\}$$

observe that $a_i(m_i + N) = a_i m_i + N$, thus

$$\sum_i a_i \cdot (m_i + N) = \underbrace{(\sum_i a_i \cdot m_i)}_{\in \mathfrak{a}M} + N \in \mathfrak{a}M + N$$

Hence,

$$\mathfrak{a}(\frac{M}{N}) = \{x + N \;\mid\; x \in \mathfrak{a}M\} = \mathfrak{a}M + N \tag{4}$$

By definition, if we take $\frac{\mathfrak{a}M+N}{N}$, then

$$\frac{\mathfrak{a}M + N}{N} = \{y + N \;\mid\; y \in \mathfrak{a}M + N\} = \mathfrak{a}M + N$$

thus every $y \in \mathfrak{a}M + N$ can be written as

$$y = x + n, \quad \text{with } x \in \mathfrak{a}M, \; n \in N$$

which comes from (4).

Thus, $y + N = (x + n) + N = x + N$, since $n \in N$ is zero in the quotient.

Hence, every element of $\frac{\mathfrak{a}M+N}{N}$ has the form

$$\frac{\mathfrak{a}M + N}{N} = \{x + N \mid x \in \mathfrak{a}M\}$$

as in (4).

Thus

$$\mathfrak{a}(\frac{M}{N}) = \mathfrak{a}M + N = \frac{\mathfrak{a}M + N}{N} \tag{5}$$

By the Collorary assumption, $M = \mathfrak{a}M + N$; quotient it by N:

$$\frac{M}{N} = \frac{\mathfrak{a}M + N}{N} \tag{6}$$

So, from (5) and (6):

$$\mathfrak{a}(\frac{M}{N}) = \mathfrak{a}M + N = \frac{\mathfrak{a}M + N}{N} = \frac{M}{N}$$

thus, $\mathfrak{a}(\frac{M}{N}) = \frac{M}{N}$.

By Nakayama's lemma AM.2.6, if $\mathfrak{a}(\frac{M}{N}) = \frac{M}{N} \overset{implies}{\Longrightarrow} \frac{M}{N} = 0$

Note that

$$\frac{M}{N} = \{m + N \mid m \in M\}$$

(the zero element in $\frac{M}{N}$ is the coset $N = 0 + N$)

Then, $\frac{M}{N} = 0$ means that the quotient has exactly one element, the zero coset N.

Thus, every coset $m+N$ equals the zero coset N, so $m-0 \in N \implies m \in N$.

Hence every $m \in M$ lies in N, ie. $\forall m \in M,\ m \in N$.

So $M \subseteq N$. But notice that by the Corollary, we had $N \subseteq M$, therefore $M = N$.

Thus, if $M = \mathfrak{a}M + N \overset{implies}{\Longrightarrow} M = N$. □

Proposition AM.2.8. Let x_i $\forall i \in [n]$ be elements of M whose images $\frac{M}{mM}$ from a basis of this vector space. Then the x_i generate M.

Proof. Let N submodule M, generated by the x_i.

Then the composite map $N \longrightarrow M \longrightarrow \frac{M}{mM}$ maps N onto $\frac{M}{mM}$, hence $N + \mathfrak{a}M = M$, which by AM.2.7 implies $N = M$. □

2.3 Sequences

Definition R.2.9.a (Exact Sequence)**.** Let a sequence of homomorphisms

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N$$

It is *exact* at M if $im(\alpha) = ker(\beta)$.

ie. $\beta \circ \alpha = 0$ and α maps surjectively to $ker(\beta)$.

Definition R.2.9.b (Short Exact Sequence (s.e.s.))**.**

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is exact $\iff$ $L \subset M$ and $N = M/L$.

Properties:

- α injective
- β surjective
- $\alpha :\ L \Longrightarrow ker\beta$
- β induces $M/\alpha(L) \longrightarrow N$

Proposition R.2.10 (Split exact sequence)**.** For the previous s.e.s., 3 equivalent conditions:

i. $\exists$ isomorphism $M \cong L \oplus N$, with

$$\alpha : m \longmapsto (m, 0)$$
$$\beta : (m, n) \longmapsto n$$

ii. $\exists$ a *section* of β, that is, a map $s : N \longrightarrow M$ such that $\beta \circ s = id_N$

iii. $\exists$ a *retraction* of α, that is, a map $r : M \longrightarrow L$ such that $r \circ \alpha = id_L$

If all i, ii, iii are satisfied, it is a split exact sequence.

Proof. Intuitively, when a s.e.s. *splits* it means that the middle module M is the direct sum of the other (outer) two modules, ie. $M = L \oplus N$.

(i to ii, iii) if $M \cong L \oplus N$ such that $\alpha :\ m \longmapsto (m, 0)$, $\beta :\ s(m, n) \longmapsto n$, we can define the maps

for ii:

$$s :\ N \longrightarrow L \oplus N$$
$$s(n) \longmapsto (0, n)$$

Then $\beta(s(n)) = \beta(0, n)$, so $\beta \circ s = id_N$.

for iii:

$$r :\ L \oplus N \longrightarrow L$$
$$r(m, n) \longmapsto m$$

Then $r(\alpha(m)) = r(m, 0)$, so $r \circ \alpha = id_L$.

(ii to i) assume $s : N \longrightarrow M$ such that $\beta \circ s = id_M$

Want to show $M \cong im(\alpha) \oplus im(s)$.

$\forall m \in M$, consider $m - s(\beta(m))$, apply β to it:
$\beta(m - s(\beta(m))) = \beta(m) - (\beta \circ s)(\beta(m)) = \beta(m) - \beta(m) = 0$

Since $ker(\beta) = im(\alpha)$, $\exists! l \in L$ such that $\alpha(l) = m - s(\beta(m))$.

Thus $m = \alpha(l) + s(\beta(m))$.

Now, suppose $x \in im(\alpha) \cap im(s)$, then $x = \alpha(l) = s(n)$, apply β to it: $\beta(\alpha(l)) = \beta(s(n)) \implies 0 = n$.

If $n = 0$, then $s(n) = 0$, so the intersection is $\{0\}$.

Define

$$\begin{aligned} \phi : L \oplus N &\longrightarrow M \\ \phi(l, n) &\longmapsto \alpha(l) + s(n) \end{aligned}$$

This isomorphism satisfies the required conditions.

(iii to i) similar to the previous one.

Overview:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, shift left, "\alpha"] & \overset{M}{\cong L \oplus N} \arrow[l, shift left, "r"] \arrow[r, shift left, "\beta"] & N \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{aligned} \alpha : l &\longmapsto (l, 0) \\ r : (m, n) &\longmapsto m \\ \alpha \circ r &= id_L \\ \beta : (l, n) &\longmapsto n \\ s : n &\longmapsto (0, n) \\ \beta \circ s &= id_N \end{aligned}$$

□

3 Noetherian rings (and modules)

Definition (Ascending Chain Condition)**.** A partially orddered set Σ has the *ascending chain condition* (a.c.c.) if every chain

$$s_1 \leq s_2 \leq \ldots \leq s_k \leq \ldots$$

eventually breaks off, that is, $s_k = s_{k+1} = \ldots$ for some k.

$\Longrightarrow$ Σ has the a.c.c. iff every non-empty subset $S \subset \Sigma$ has a maximal element.

if $\neq S \subset \Sigma$ does not have a maximal element, choose $s_1 \in S$, and for each s_k, an element s_{k+1} with $s_k < s_{k+1}$, thus contradicting the a.c.c.

3.1 Noetherian rings and modules

Definition R.3.2 (Noetherian ring)**.** Let A a ring; 3 equivalent conditions:

i. the set Σ of ideals of A has the a.c.c.; in other words, every increasing chain of ideals
$$I_1 \subset I_2 \subset \ldots \subset I_k \subset \ldots$$
eventually stops, that is $I_k = I_{k+1} = \ldots$ for some k.

ii. every nonempty set S of ideals has a maximal element

iii. every ideal $I \subset A$ is finitely generated

If these conditions hold, then A is *Noetherian.*

Proof. TODO □

Definition R.3.4.D (Noetherian modules)**.** An A-module M is Noetherian if the submoles of M have the a.c.c.,
that is, ay increasing chain
$$M_1 \subset M_2 \subset \ldots \subset M_k \subset \ldots$$
of submodules eventually stops.

As in with rings, it is equivalent to say that

i. any nonempty set of modulesof M has a maximal element

ii. every submodule of M is finite

Proposition R.3.4.P. Let $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ be a s.e.s. (split exact sequence, R.2.10).

Then, M is Noetherian $\Longleftrightarrow$ L and N are Noetherian.

Proof. $\Longrightarrow$: trivial, since ascending chains of submodules in L and N correspond one-to-one to certain chains in M.

$\Longleftarrow$: suppose $M_1 \subset M_2 \subset \ldots \subset M_k \subset \ldots$ is an increasing chain of submodules of M.

Then identifying $\alpha(L)$ with L and taking intersection gives a chain
$$L \cap M_1 \subset L \cap M_2 \subset \ldots \subset L \cap M_k \subset \ldots$$
of submodules of L, and applying β gives a chain
$$\beta(M_1) \subset \beta(M_2) \subset \ldots \beta(M_k) \subset \ldots$$

of submodules of N.

Each of these two chains eventually stop, by the assumption on L and N, so that we only need to prove the following lemma which completes the proof. □

Lemma R.3.4.L. for submodules $M_1 \subset M_2 \subset M$,

$$L \cap M_1 = L \cap M_2 \text{ and } \beta(M_1) = \beta(M_2) \implies M_1 = M_2$$

Proof. if $m \in M_2$, then $\beta(m) \in \beta(M_1) = \beta(M_2)$, so that there is an $n \in M_1$ such that $\beta(m) = \beta(n)$.

Then $\beta(m - n) = 0$, so that

$$m - n \in M_2 \cap ker(\beta) = M_1 \cap ker(\beta)$$

Hence $m \in M_1$, thus $M_1 = M_2$. □

Corollary R.3.5 (Properties of Noetherian modules)**.** i. if $\forall i \in [r]$, M_i are Noetherian modules, then $\bigoplus_{i=1}^{r} M_i$ is Noetherian.

ii. if A a Noetherian ring, then an A-module M is Noetherian iff it is finite over A.

iii. if A a Noetherian ring, M a finite module, then any submodule $N \subset M$ is again finite.

iv. if A a Noetherian ring, and $\psi : A \longrightarrow B$ a ring homomorphism such that B is a finite A-module, then B is a Noetherian ring.

Proof. i. a direct sum $M_1 \oplus M_2$ is a particular case of an exact sequence.

Then, Proposition R.3.4.P proves this statement when $r = 2$. The case $r > 2$ follows by induction.

ii. if M finite, then $\exists$ surjective homomorphism

$$A^r \longrightarrow M \longrightarrow 0$$

for some r, so that M is a quotient

$$M \cong A^r/N$$

for some submodule $N \subset A^r$.

A^r is a Noetherian module by i., so M is Noetherian due Proposition R.3.4.P.

Conversely, M Noetherian implies M finite.

item as in previous implications:
M finite and A Noetherian $\Longrightarrow$ M is Noetherian,
$\Longrightarrow$ since $N \subseteq M$, then N is Noetherian too
$\Longrightarrow$ which implies that N is a finite A-module.

iii. B is Noetherian as an A-module; but ideals of B are submodules of B as an A-submodule, so that B is a Noetherian ring.

□

3.2 Hilbert basis

Theorem R.3.6 (Hilbert basis theorem)**.** If A a Noetherian ring, then so is the polynomial ring $A[x]$.

Proof. Prove that any ideal $I \subset A[x]$ is fingen.

Define auxiliary sets $J_n \subset A$ by

$$J_n = \{a \in A \mid \exists f \in I \text{ s.th. } f = ax^n + b_{n-1}x^{n-1} + \dots b_0\}$$

ie. J_n is the set of leading coefficients of I of degree n.

J_n is an ideal, since I is an ideal.

$J_n \subset J_{n+1}$, since for $f \in I$ also $xf \in I$.

Therefore $J_1 \subset J_2 \subset \dots \subset J_k \subset \dots$ is an increasing chain of ideals. Using the assumption that A is Noetherian, deduce that $J_n = J_{n+1}$ for some n.

For each $m \leq n$, $J_m \subset A$ is fingen, ie.

$$J_m = (a_{m,1}, \dots a_{m,r_m})$$

By definition of J_m, for each $a_{m,j}$ with $1 \leq j \leq r_m$, $\exists$ a polynomial $f_{m,j} \in I$ of degree m having the leading coefficient $a_{m,j}$.

$$\implies \quad \{f_{m,j}\}_{m<n; 1 \leq j \leq r_m}$$

the set of elements of I.

Claim: this finite set ($\{f_{m,j}\}$) generates I.

$\forall f \in I$, if $\deg f = m$, then its leading coefficient is $a \in J_m$,

hence if $m \geq n$, then $a \in J_m = J_n$, so that

$$a = \sum b_i a_{n,i} \quad \text{with } b_i \in A$$

and

$$f - \sum b_i X^{m-n} \cdot f_{n,i}$$

has degree $< m$.

Similarly, if $m \leq n$, then $a \in J_m$, so that

$$a = \sum b_i a_{m,i} \quad \text{with } b_i \in A$$

and

$$f - \sum b_i f_{n,i}$$

has degree $< m$.

By induction on m, f can be written as a linear combination of finitely many elements.

Thus, any ideal of $A[x]$ is finitely generated. □

Corollary R.3.6.C. if A a Noetherian ring, and $\psi : A \longrightarrow B$ a ring homomorphism such that B is a fingen extension ring of $\psi(A)$, then B is Noetherian.

In particular, any fingen algebra over $\mathbb{Z}$ or over a field K is Noetherian.

Proof. the assumption is that B is a quotient of a polynomial ring,

$$B \cong A[x_1, \dots, x_n]/I$$

for some ideal I.

By the Hilbert basis theorem R.3.6 and induction,
A being Noetherian implies that $A[x_1, \dots, x_n]$ is Noetherian.

And by Corollary R.3.5(iv),
$A[x_1, \dots, x_n]$ being Noetherian implies that $A[x_1, \dots, x_n]/I$ is Noetherian. □

4 Finite ring extensions and Noether normalization

4.1 A-algebras and integral domains

Definition (A-algebra)**.** An A-algebra is a ring B with a ring homomorphism $\psi : A \longrightarrow B$.

B is an A-module with multiplication defined by $\psi(a) \cdot b \quad (a \in A, b \in B)$.

When $A \subset B$, B is an extenaion ring of A; denoted $\psi(A) = A' \subset B$.

Definition R.4.1. Let B be an A-algebra.

i. B is a *finite* A-algebra (*finite over* A) if it is finite as an A-module.

ii. $y \in B$ is *integral over* A if $\exists$ a monic polynomial

$$f(Y) = Y^n + a_{n-1}Y^{n-1} + \ldots + a_0 \ \in A'[Y]$$

such that $f(y) = 0$:

$$f(y) = y^n + a_{n-1}y^{n-1} + \ldots + a_0 = 0$$

The algebra B is *integral over* A if $\forall$ $b \in B$ is integral.

Proposition R.4.2. Let $\psi : A \longrightarrow B$ be an A-algebra, and $y \in B$. Three equivalent conditions:

i. y is integral over A

ii. subring $A'[y] \subset B$ generated by $A' = \psi(A)$ and y is finite over A

iii. $\exists$ an A-subalgebra $C \subset B$ such that $A'[y] \subset C$ and C is finite over A

Notes: A' is the image of A in B, ie. $A' = \psi(A)$.
$A'[y]$ is the smallest subring of B containing both coefficients from A and the element y.

Proof. .

(i to ii): since y integral over $A \implies$ by R.4.1 (ii), y satisfies

$$f(y) = y^n + a_{n-1}y^{n-1} + \ldots + a_0 = 0$$

So any power y^k $(k \geq n)$ can be expressed in terms of $\{1, y, y^2, \ldots, y^{n-1}\}$.

Thus the set $\{1, y, y^2, \ldots, y^{n-1}\}$ spans $A'[y]$ as an A-module.

(iii to i): since $A'[y] \subset C \implies y \in C$
since C finite over A, C has finite generators $\{c_1, \ldots, c_n\}$ such that $C = A \cdot c_1 + A \cdot c_2 + \ldots + A \cdot c_n$

Thus $y \cdot c_i \in C$,

$$y \cdot c_i = \sum_{j=1}^{n} a_{ij} c_j$$

with $a_{ij} \in A$.

By the Cayley-Hamilton theorem (AM.2.4),

$$y^n + a_{n-1}y^{n-1} + \ldots + a_1 y + a_0 = 0$$

Therefore, y is integral (by R.4.1 (ii)).

□

Proposition R.4.3 (Tower Laws)**.** Let B be an A-algebra.

a. Transitivity of finiteness: if $A \subset B \subset C$ are extension rings such that C is a finite B-algebra and B a finite A-algebra,
then C is finite over A.

b. Finiteness of generated algebras: if $y_1, \ldots, y_m \in B$ are integral over A, then $A[y_1, \ldots, y_m]$ is finite over A.
In particular, every $f \in A[y_1, \ldots, y_m]$ is integral over A.

c. Transitivity of integrality: if $A \subset B \subset C$ with C integral over B, and B integral over A,
then C is integral over A.

d. Integral closure as a subring: the subset

$$\tilde{A} = \{y \in B \mid y \text{ is integral over } A\} \subset B$$

is a subring of B.

Moreover, if $y \in B$ is integral over $\tilde{A}$ then $y \in \tilde{A}$, so that $\tilde{\tilde{A}} = \tilde{A}$.

Proof. .

a. if $\{\beta_1, \ldots, \beta_n\}$ generate B as an A-module and $\{\gamma_1, \ldots, \gamma_n\}$ generate C as an B-module,
then the set of products $\{\beta_i \gamma_j\}$ generates C as an A-module.

Since there are $n \times m$ generators (ie. finite), C is finite over A.

b. proof by induction:

base case: if y_1 integral over $A \implies$ it satisfies a monic polynomial.

Thus $A[y_1]$ is generated as an A-module by $\{1, y_1, y_1^2, \ldots, y_1^{n-1}\}$, making it a finite A-algebra.

inductive step: let $R_k = A[y_1, \ldots, y_k]$. Assume R_k is finite over A.

Since y_{k+1} is integral over $A \implies$ it is also integral over R_k.

Thus $R_{k+1} = R_k[y_{k+1}]$ is finite over R_k.

Applying part (a) (transitivity of finiteness), if R_{k+1} is finite over R_k and R_k finite over A, then R_{k+1} is finite over A.

Consequence: since any $f \in A[y_1, \ldots, y_m]$ belongs to a finite A-algebra, f must be integral over A (since an element is integral iff it is contained in a finite extension).

c. let $x \in C$, since x integral over B, it satisfies:

$$x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 = 0, \qquad b_i \in B$$

Let $B'' = A[b_0, b_1, \ldots, b_{n-1}]$. Since each $b_i \in B$ and B is integral over A $\implies$ each b_i is integral over A.

Since all b_i are integral over $B' \implies B'[x]$ is a finite B'-algebra.

By part (a) (transitivity of finiteness), $B'[x]$ is a finite A-algebra.

Therefore, x is integral over A.

d. I. subring:
let $x, y \in \tilde{A}$. Want to show $x + y, xy \in \tilde{A}$:
by part (b), the algebra $A[x, y]$ is finite over A.
Since $x + y, xy \in A[x, y]$, they are integral over A.
Thus $x + y, xy \in \tilde{A}$, since $\tilde{A} = \{b \in B \mid b \text{ integral over } A\}$.

II. idempotence
let $z \in B$ be integral over $\tilde{A}$
we have a chain $A \subseteq \tilde{A} \subseteq \tilde{A}[x]$.
By definition, $\tilde{A}$ is integral over A, and z is integral over $\tilde{A}$
thus by part (c), z is integral over A.
Therefore, $z \in \tilde{A}$.

□

Lemma 4.3.Aux (Integrality implies finiteness)**.** If y integral over A then $A[y]$ is finite over A.

This extends on point (b) from the previous proposition R.4.3.

Proof. Suppose y is integral over A. By definition $\exists \;\; f \in A[T]$, with f monic, such that $f(y) = 0$.

Let $deg(f) = d$, so that for $f(y) = 0$ we have

$$y^d + a_{d-1}y^{d-1} + \ldots + a_1 y + a_0 = 0 \qquad a_i \in A$$

Since it is monic (leading coefficient is 1), we can rearrange it to isolate the highest power:

$$y^d = -(a_{d-1}y^{d-1} + \ldots + a_1 y + a_0) \tag{7}$$

Thus y^d can be written using lower powers of y with coefficients in A.

Consider any element $p \in A[y]$, $p = c_m y^m + c_{m-1}y^{m-1} + \ldots + c_0$.

if $m < d$, leave it as it is.
if $m \geq d$, use the monic equation (7) to replace y^d with lower powers.

Repeating this process, can reduce any power of y down to a linear combination of $\{1, y, y^2, \ldots, y^{d-1}\}$.

Thus every element in $A[y]$ can be expressed as

$$\lambda_{d-1}y^{d-1} + \ldots + \lambda_2 y^2 + \lambda_1 y + \lambda_0 \cdot 1 \qquad \lambda_i \in A$$

Henceforth, the set $\{1, y, y^2, \ldots, y^{d-1}\}$ generates $A[y]$ as a finite A-module.

□

Definition 4.4 (Integral closure)**.** Given the ring $\tilde{A}$ from R.4.3.(d), ie. $\tilde{A} = \{y \in B \mid y \text{ integral over } A\} \subset B$, $\tilde{A}$ is the *integral closure* of A in B.

If $A = \tilde{A}$, then A is *integrally closed* in B.

An integral domain A is *normal* if it is *integrally closed in its field of fractions*, that is if

$$A = \tilde{A} \subset K = Frac(A)$$

For any integral domain A, the integral closure of A in its field of fractions $K = Frac(A)$ is also called the *normalization* of A.

4.2 Noether normalization

Definition 4.6 (Algebraically independent)**.** $y_1, \dots, y_n \in A$ are *algebraically independent* over K if the natural surjection $K[Y_1, \dots, Y_n] \longrightarrow K[y_1, \dots, y_n]$ is an isomorphism.

$\implies \nexists\ F(y_1, \dots, y_n) = 0$ (F nonzero) with coefficients in K.

Recall: a K-algebra A is fingen over K if $A = K[y_1, \dots, y_n]$ for some finite set $y_1, \dots, y_n$.

Lemma R.4.6.L. Let $A = K[y_1, \dots, y_n]$ and $0 \neq F \in K[Y_1, \dots, Y_n]$ such that $F(y_1, \dots, y_n) = 0$.

Then $\exists\ y_1^*, \dots, y_{n-1}^* \in A$ such that y_n is integral over

$$A^* = K[y_1^*, \dots, y_{n-1}^*] \text{ and } A = A^*[y_n]$$

Proof. Set $y_i^* = y_i - y_n^{r_i}$ for $i \in [n-1]$ and $r_1, \dots, r_{n-1} \geq 1 \in \mathbb{Z}$.
(ie. $y_i = y_i^* + y_n^{r_i}$)

Define $G \in A$ by

$$G(y_1^*, \dots, y_{n-1}^*, y_n) = F(y_i^* + y_n^{r_i}, y_n) = 0$$

viewed as a relation for y_n over $K[y_1^*, \dots, y_{n-1}^*]$.

Since F is a polynomial in $y_1, \dots, y_{n-1}^*$, can write it as a sum of monomials

$$F = \sum_m a_m y^m = \sum_m a_m \prod y_i^{m_i}$$

where $m = (m_1, \dots, m_n)$ and each $a_m \neq 0$.

Therefore,

$$G = \sum a_m \prod (y_i^* + y_n^{r_i})^{m_i}$$

which when expanding out, each summand $a_m \prod (y_i^* + y_n^{r_i})^{m_i}$ has a unique term of highest order in y_n, namely $a_m y_n^{(\sum r_i m_i)}$.

Suppose we can arrange so that

$$m \neq m' \implies \sum r_i m_i \neq \sum r_i m_i'$$

Then $max\{\sum r_i m_i \mid m \text{ s.th. } a_m \neq 0\}$ is achieved in only one summand, so that here is no cancellation; thus the highest order term in G is $a_m y_n^{(\sum r_i m_i)}$ (ie. a_m times a pure power of y_n). □

Theorem R.4.6 (Noether normalization lemma)**.** Let K a field, A a fingen K-algebra.

Then $\exists\ z_1, \dots, z_m \in A$ such that

i. $z_1, \dots, z_m$ are algebraically independent over K

ii. A is finite over $B = K[z_1, \dots, z_m]$

That is, a fingen extension $K \subset A$ can be written as a composite

$$K \subset B = K[z_1, \dots, z_m] \subset A$$

where $K \subset B$ is a polynomial extension, and $B \subset A$ is finite.

Proof. Let $y_1, \dots, y_n$ be generators of $A = K[y_1, \dots, y_n]$.
if $n = 0$, nothing to prove since A is generated by 0 elements $\implies A = K$, and K is finite.
if $n > 0$ we have two cases:

- $y_1, \dots, y_n$ are algebraically independent over K, then by definition 4.6 $A \cong K[y_1, \dots, y_n]$, so that A is a finite module over itself, with $m = n$.

- $y_1, \dots, y_n$ are algebraically dependent over K,

$$\exists\ 0 \neq f \in K[y_1, \dots, y_n] \text{ s.th } f(y_1, \dots, y_n) = 0$$

Want f to be *monic*, so that y_n is integral over new defined variables $y_1^*, \dots, y_{n-1}^*$. In other words, want some polynomial like

$$y_n^d + a_{d-1} y_n^{d-1} + \dots + a_1 y_n + a_0 = 0 \qquad a_i \in K[y_1^*, \dots, y_{n-1}^*]$$

ie. monic, so that by definition (R.4.1), y_n is integral over $K[y_1^*, \dots, y_{n-1}^*]$.
$\longrightarrow$ Change variables so that f becomes monic in one of the variables (y_n); this allows to express one generator (y_n) as an integral element over the others.

Following from Lemma R.4.6.L, define the new variables $y_1^*, \dots, y_{n-1}^* \in A$ such that y_n is integral over

$$A^* = K[y_1^*, \dots, y_{n-1}^*], \text{ and } A = A^*[y_n]$$

Setting $y_i^* = y_i - y_n^{r_i}$, so that $y_i = y_i^* + y_n^{r_i}$ $\forall i \in [n-1]$, $r_1, \dots, r_{n-1} \geq 1 \in \mathbb{Z}$.
Use those new variables at $f(y_1, \dots, y_n) = 0$:

$$f(y_1^* + y_n^{r_1}, y_2^* + y_n^{r_2}, \dots, y_{n-1}^* + y_n^{r_{n-1}}, y_n) = 0$$

Then the highest power of y_n in each term of f will look like $y_n^{(\sum a_i r_i)}$, and with r_i growing fast enough we ensure that each monomial in f produces a unique power of y_n.

Then we have $c \cdot y_n^D +$ (terms with lower powers of y_n) $= 0$ with $c \in K \setminus \{0\}$. So that dividing by c we get the shape $y_n^D + \dots = 0$, thus y_n is integral over $A^* = K[y_1^*, \dots, y_{n-1}^*]$.

Induction:
Since y_n integral over $A^* \implies A = A^*[y_n]$ is finite over $A^* = K[y_1^*, \dots, y_{n-1}^*]$ (by 4.3.Aux).

By inductive hypothesis on A^*, $\exists\, z_1, \ldots, z_m \in A^*$ algebraically independent over K and with A^* finite over $B = K[z_1, \ldots, z_m]$.

Therefore, each step of $B \subset A^* \subset A^*[y_n] = A$ is finite, and A is finite over B as required. □

Example . $A = K[X,Y]/(XY-1)$. Y is algebraic over $K[X]$, but not integral over $K[Y]$.

This corresponds to the fact that the hyperbola $XY = 1$ has the line $X = 0$ as an asymptotic line (so that its projection to the X-axis misses a root over $X = 0$).

Take $X' = X - \epsilon Y$ as the element of A instead of X; then the relation becomes $(X' + \epsilon Y)Y = 1$, monic in Y if $\epsilon \neq 0$.

This corresponds geometrically to tilting the hyperbola a little before projecting, so that no longer has a vertical asymptotic line.

4.3 Weak Nullstellensatz

Proposition R.4.9. let $A \subset B$ be an integral extension of integral domain, then A is a field $\iff$ B is a field.

Proof. $\Longrightarrow$:
let $0 \neq x \in B$, then $\exists\ \ x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0 \quad a_i \in A$, monic.

Since A is a field, $\exists$ inverse, observe that:

$$\begin{aligned} x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0 \\ x(x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_1) = -a_0 \\ -a_0^{-1}(x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_1) = x^{-1} \in B \end{aligned}$$

thus there exists inverse in B, so B is a field too.

$\Longleftarrow$:
if B is a field and $0 \neq x \in A$, then $x^{-1} \in B$, so x^{-1} is integral over A.

So there is a relation of the form

$$(x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \ldots + a_0 = 0$$

Therefore

$$\begin{aligned} (x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \ldots + a_0 &= 0 \\ (x^{-1})^n &= -a_{n-1}(x^{-1})^{n-1} - \ldots - a_0 \\ x^{-n} &= -a_{n-1}x^{-n+1} - \ldots - a_0 \ \ \text{(mult by } x^{n-1}\text{)} \\ x^{-n+(n-1)} &= -a_{n-1}x^{-n+1+(n-1)} - \ldots - a_0 x^{n-1} \\ x^{-1} &= -a_{n-1} - \ldots - a_0 x^{n-1} \in A \end{aligned}$$

thus there exists inverse in A, so A is a field too. □

Theorem R.4.10 (Weak Nullstellensatz - Zariski's lemma)**.** let k a field, K a k-algebra which

1. is finitely generated as a k-algebra
2. is a field

Then K is algebraic over k, so that $k \subset K$ is a finite field extension. That is, $[K : k] < \infty$.

Proof. $K = k[z_1, \dots, z_m]$ a field; want to show that K is algebraic over k.

Since K is a fingen k-algebra, by Noether normalization lemma (R.4.6), $\exists\ z_1, \dots, z_m \in K$ such that

- are algebraically independent
- K is integral over the polynomial ring $A = k[z_1, \dots, z_m]$ (which by 4.3.Aux is finite)

Now we're at the situation of R.4.9:
$A \subset K$ is integral, K is a field $\implies$ therefore A is a field.

Since $z_1, \dots, z_m \in K$ are algebraically independent,
$\implies$ $A = k[z_1, \dots, z_m]$ is a polynomial ring in m indeterminates, and this is a field only if $m = 0$
(since in $k[z_1]$ the element z_1 is not invertible, since $1/z_1$ is a rational function, not a polynomial).
So $A = k$; which by Noether normalization we saw that K is integral over $A = k$, and by 4.3.Aux that it is finite, thus K is finite over k, ie. $[K : k], \infty$, and K is algebraic over k. □

5 Nullstellensatz

Note: for k a field, $k[X_1, \dots, X_n]$, m maximal ideal; the residue field $K = k[X_1, \dots, X_n]/m$ satisfies the Zariski's lemma (R.4.10), thus K is a finite algebraic extension of k.

Corollary 5.2. k algebraically closed. Then every maximal ideal of $A = k[X_1, \dots, X_n]$ is of the form

$$m = (X_1 - a_1, \dots, X_n - a_n), \quad a_i \in k$$

The map $k[X_1, \dots, X_n] \longrightarrow k[X_1, \dots, X_n]/m = k$ is the natural evaluation map $f(X_1, \dots, X_n) \longmapsto f(a_1, \dots, a_n)$.

Thus

$$\begin{aligned} k^n &\longleftrightarrow m - \mathit{Spec}\ A \\ (a_1, \dots, a_n) &\longleftrightarrow f(a_1, \dots, a_n) \end{aligned}$$

Proof. let $m \subset k[X_1, \dots, X_n]$ be a maximal ideal.

By fundamental property of maximal ideals, $K = A/m$ is a field.

Since A is a fingen k-algebra (generated by $X_1, \dots, X_n$), then $K = A/m$ is also a fingen k-algebra, generated by residues $x_i' = x_i + m$.

By Zariski's lemma (R.4.10), $K = A/m$ is algeraic over k.

Since by hypothesis k is algebraically closed, it has no proper algebraic extensions

$\implies K = k \implies k \cong A/m$.

So, $\forall x_i \in k$, its image in the quotient field A/m must be an element of k.

$$\implies x_i' = a_i \in k,\ \forall i \in [n]$$

$$\implies x_i - a_i \in m$$

The ideal generated by these terms is a subset of m:

$$J = (X_1 - a_1, \dots, X_n - a_n) \subseteq m$$

Since J is the kernetl of the evaluation map at point $(a_1, \dots, a_n)$, then J is a maximal ideal. Together with $J \subseteq m$, then we have $J = m$, ie.

$$m = (X_1 - a_1, \dots, X_n - a_n)$$

Let

$$\begin{aligned} \psi : k[X_1, \dots, X_n] &\longrightarrow k[X_1, \dots, X_n]/m \\ \psi : x_i &\longmapsto a_i \end{aligned}$$

Since ψ is a k-algebra homomorphism, then $\forall f \in A$:

$$\psi(f(X_1, \dots, X_n)) = f(\psi(x_1), \dots, \psi(x_n)) = f(a_1, \dots, a_n)$$

Thus there is a one-to-one correspondence:
points in k^n $\longleftrightarrow$ $m - \mathit{Spec}\ A$ (maximal ideals in $k[X_1, \dots, X_n]$
$(a_1, \dots, a_n)$ $\longleftrightarrow$ $(X_1 - a_1, \dots, X_n - a_n)$ □

5.1 Variety

Definition 5.3 (Variety)**.** A *variety* $V \subset k^n$:

$$V = V(J) = \{P = (a_1, \ldots, a_n) \in k^n | f(P) = 0 \; \forall \; f \in J\}$$

→ V is defined by $f_1(P) = \ldots = f_m(P) = 0$

→ V is defined as the simultaneous solutions of a number of polynomial equations.

Proposition 5.3. *k an algebraically closed field, and $A = k[X_1, \ldots, X_n]$ a fingen k-algebra of the form $A = k[X_1, \ldots, X_n]/J$, where J is an ideal of $k[X_1, \ldots, X_n]$. (notation: $x_i = X_i \pmod{J}$)*

Then every maximal ideal of A is of the form

$$(x_1 - a_1, \ldots, x_n - a_n)$$

for some point $(a_1, \ldots, a_n) \in V(J)$.

Therefore, $\exists$ a one-to-one correspondence

$$\begin{aligned} V(X) &\longleftrightarrow m - \mathit{Spec}\ A \\ \text{given by} \quad (a_1, \ldots, a_n) &\longleftrightarrow (x_1 - a_1, \ldots, x_n - a_n) \end{aligned}$$

Proof. the ideals of A are given by ideals of $k[X_1, \ldots, X_n]$ containing J, since for $Q = R/I$, $\exists$ one-to-one correspondence between ideals of Q and ideals of R that contain I, ie.

$$\begin{aligned} m \subset A \longleftrightarrow & m' \subseteq k[X_1, \ldots, X_n] \\ & \text{s.th. } J \subseteq m' \end{aligned}$$

Thus, every maximal ideal of A is of the form

$$\underbrace{(x_1 - a_1, \ldots, x_n - a_n)}_{i.} \text{ such that } \underbrace{J \subset (X_1 - a_1, \ldots, X_n - a_n)}_{ii.}$$

i. Since k is algebraically closed, the maximal ideals of $k[X_1, \ldots, X_n]$ look like $m = (X_1 - a_1, \ldots, X_n - a_n)$ for some point $(a_1, \ldots, a_n) \in k^n$.

   Which when projected to the quotient ring A, $X_i \longmapsto x_i$ (residue class), giving $(x_1 - a_1, \ldots, x_n - a_n)$.

ii. for m to exist in A, the corresponding m' must contain J; since if it didn't contain J it wouldn't "survive" the quotient process.

However, since $(X_1 - a_1, \ldots, X_n - a_n)$ is the kernel of the evaluation map $f \longmapsto f(a_1, \ldots, a_n)$

$\Longrightarrow$ means that $m' = (X_1 - a_1, \ldots, X_n - a_n)$ consists of all polynomials that vanish at point $P = (a_1, \ldots, a_n)$.

If $J \subseteq m'$, then $\forall\ f \in J$ must vanish at P.

By definition, the set of points where all polynomials in J vanish is the *variety*, $V(J)$.

Thus,
every maximal ideal in A corresponds to a point $(a_1, \dots, a_n) \in k^n$, ie.

$$m - Spec\ A \longleftrightarrow k^n$$

The condition that the ideal belongs to the quotient ring $A = k[X_1, \dots, X_n]/J$ forces that point to lie in $V(J)$, so

$$m - Spec\ A \longleftrightarrow V(J)$$
$$\text{maximal spectrum} \longleftrightarrow \text{variety}$$

□

Proposition 5.5 (Correspondeces V and I)**.** A variety $X \subset k^n$ is by definition $X = V(J)$ (J an ideal of $k[X_1, \dots, X_n]$).

So V gives a map:

$$\{\text{ideals of } k[X_1, \dots, X_n]\} \xrightarrow{V} \{\text{subsets } X \text{ of } k^n\}$$

correspondence going the other way:

$$\{\text{subsets } X \text{ of } k^n\} \xrightarrow{I} \{\text{ideals of } k[X_1, \dots, X_n]\}$$

defined by taking a subset $X \subset k^n$ into the ideal

$$I(X) = \{f \in k[X_1, \dots, X_n] | f(P) = 0\ \forall\ P \in X\}$$

$V,\ I$ satisfy reverse inclusions:

$$J \subset J' \implies V(J) \supset V(J') \quad \text{and} \quad X \subset Y \implies I(X) \supset I(Y)$$

5.2 Nullstellensatz

Theorem 5.6 (Nullstellensatz)**.** Let k algebraically closed field.

a. if $J \subsetneq k[X_1, \dots, X_n]$ then $V(J) \neq \emptyset$

b. $I(V(J)) = radJ$, in other words, for $f \in k[X_1, \dots, X_n]$,

$$f(P) = 0\ \forall\ P \in V \iff f^n \in J \text{ for some } n.$$

Proof. a. if $J \subsetneq k[X_1, \dots, X_n]$ then $V(J) \neq \emptyset$:
Let $m \subset k[X_1, \dots, X_n]$ be a maximal ideal.
Then $L = k[X_1, \dots, X_n]/m$ is a field (by TODO ref).

By Zariski's lemma (R.4.10), since L is generated as a k-algebra by the images of the variables x_i, and k is algebraically closed.

Then the only algebraic extension of k is k itself. Thus $L \cong k$.

Then $\exists$ a surjective homomorphism $\psi : k[X_1, \dots, X_n] \longrightarrow k$.

Let $a_i = \psi(x_i)$. Then $x_i - a \in ker(\psi) = m \ \forall \ i$.

Since the ideal $(X_1 - a_1, \dots, X_n - a_n)$ is maximal and contained in m, they must be equal, ie. $m = (X_1 - a_1, \dots, X_n - a_n)$.

Therefore, $P = (a_1, \dots, a_n) \in k^n$ is a zero for every polynomial in m.

Since $J \subseteq m$, P is also a zero for every polynomial in J.
$\implies$ thus $P \in V(J)$, and thus $V(J) \neq \emptyset$.

b. $I(V(J)) = radJ$:

$$I(V(J)) = radJ$$
$$\text{vanishing ideal of a variety} = \text{radical of the ideal defining the variety}$$

where $rad\ J = \{f \in R \mid f^n \in J \text{ for some } n > 0\}$.

Want to show that if a polynomial vanishes at all points where $g_1, \dots, g_m$ vanish, then $f \in rad(g_1, \dots, g_m)$.

Consider the ring $k[X_1, \dots, X_n, Y]$ and the ideal J' generated by $\{g_1, \dots, g_m, 1 - Yf\}$

Suppose there is a point $(a_1, \dots, a_n, a_{n+1})$ that is a zero of J'. ie.

$$\exists \ (a_1, \dots, a_n, a_{n+1}) \in V(J')$$

Since $g_i(a) = 0$, our hypothesis says $f(a) = 0$. However, the last generator $(1 - Yf)$ requires

$$1 - a_{n+1}f(a) = 0 \implies \text{implies } 1 - a_{n+1} \cdot 0 = 0 \implies 1 - 0 = 0$$

a contradiction.

Therefore, $V(J') = \emptyset$.

Since $V(J') = \emptyset$, by the Weak Nullstellensatz/Zariski (R.4.10),
if $V(J') = \emptyset$ then $J' = (1)$, so $1 \in J' = (1)$.

Every element in an ideal is a linear combination of its generators: J' is generated by $\{g_1, \dots, g_m, 1 - Yf\}$

$$\implies \forall j \in J', \quad j = (\sum (\text{polynomial})\ g_i) + (\text{polynomial}) \cdot (1 - Yf)$$

which, since $1 \in J'$,

$$1 = \left(\sum_{i=1}^{m} p_i(X,Y) g_i(X)\right) + q(X,Y) \cdot (1 - Yf(X))$$

substitute $Y = 1/f$,

$$1 = \left(\sum_{i=1}^{m} p_i(X, \frac{1}{f}) g_i(X)\right) + q(X, \frac{1}{f}) \cdot \underbrace{(1 - \frac{1}{f} f(X))}_{0}$$

thus

$$1 = \sum_{i=1}^{m} p_i(X, \frac{1}{f}) g_i(X)$$

multiply by f^n,

$$f^n = \sum_{i=1}^{m} A_i(X) g_i(X)$$

thus f^n is a linear combination of g_i.
Thus $f^n \in J$, so $f \in rad\ J$.

□

5.3 Irreducible varieties

Definition 5.7 (Irreducible variety)**.** a variety $X \subset k^n$ is *irreducible* if it is nonempty and not the union of two proper subvarieties; that is, if

$$X = X_1 \cup X_2 \text{ for varieties } X_1, X_2 \implies X = X_1 \text{ or } X_2$$

Proposition 5.7. a variety X is irreducible iff $I(X)$ is prime.

Proof. set $I = I(X)$.
if I not prime, then $f, g \in A \setminus I$ be such that $fg \in I$.
Define new ideals

$$J_1 = (I, f) \text{ and } J_2 = (I, g)$$

Then, since $f \notin I(X)$, it follows that $V(J_1) \subsetneq X$
$\implies$ so $X = V(J_1) \cup V(J_2)$ is reducible.
The converse is similar. □

Corollary 5.8. let k algebraically closed field. Then V and I induce one-to-one correspondences

$$\{\text{radical ideals } J \text{ of } k[X_1,\dots,X_n]\} \longleftrightarrow \{\text{varieties } X \subset k^n\}$$
$$\{\text{prime ideals } P \text{ of } k[X_1,\dots,X_n]\} \longleftrightarrow \{\text{irreducible varieties } X \subset k^n\}$$

Therefore,

$$\mathit{Spec}\ k[X_1,\dots,X_n] = \{\text{irreducible varieties } X \subset k^n\}$$

Proposition 5.8. let $A = k[x_1,\dots,x_n]$ a fingen k-algebra (k an algebraically closed field).

Write J for the ideal of relations holding between $x_1,\dots,x_n$, so that $A = k[X_1,\dots,X_n]/J$.

Then there is a one-to-one correspondence

$$\mathit{Spec}\ A \longleftrightarrow \{\text{irreducible subvarieties } X \subset V(J)\}$$

Proof. By definition, *Spec* $A = \{P \mid P \subset A \text{ is prime ideal}\}$.

By Corollary 5.8:

$$\{\text{prime ideals } P \text{ of } k[X_1,\dots,X_n]\} \longleftrightarrow \{\text{irreducible varieties } X \subset k^n\}$$

About varieties:

- $I(X)$ in $R = k[X_1,\dots,X_n]$ is the ideal of the variety X
  ie. the set of all polynomials that vanish on every point of X.
- $I(X)$ in $A = k[X_1,\dots,X_n]/J$, we're not looking at all possible polynomials but at the residue classes.

If P is prime in $A \implies$ it must correspond to some prime ideal $\mathfrak{P}$ in R (also $J \subset R$).

Then from Nullstellensatz (5.6), every prime ideal $\mathfrak{P}$ in R is the ideal of some irreducible variety X.

$\implies \mathfrak{P} = I(X)$.

Since we're restricted to the ring $A = k[X_1,\dots,X_n]/J$, the ideal P are the elements of $I(X)$ viewed through the lens of the quotient

$$\implies P = \{f + J \mid f \in I(X)\} \quad = I(X) \bmod J$$

Now, J is the set of equations defining our "universe" $V(J)$.

Since $A = R/J \implies$ we thus have $J \subseteq R$.

Also we have a correspondence between $A = R/J$ and R.

Let $\mathfrak{P}$ be the preimage of P in $R = k[X_1,\dots,X_n]$, ie.

$$\begin{aligned} R &\longrightarrow A = R/J \\ \mathfrak{P} &\longmapsto P \\ I(X) &\longmapsto I(X) \bmod J \end{aligned}$$

$\implies$ thus $J \subseteq \mathfrak{P}$.

Now, $J\mathfrak{P} = I(X)$;

by 5.5, the set of points where $\mathfrak{P}$ vanishes must be inside the set of points where J vanishes, so

$$V(J) \supseteq V(\mathfrak{P}) = V(I(X)) = X$$

$\implies \quad X \subseteq V(J)$, ie. the irreducible variety X must be a subvariety of $V(J)$.

Therefore,

$$\mathfrak{P} \in Spec\ A \longleftrightarrow X \subseteq V(J)$$

where $X = V(\mathfrak{P})$. □

6 Rings of fractions $S^{-1}A$ and localization

6.1 Rings of fractions $S^{-1}A$

Definition 6.1 (ring of fractions)**.** let A a ring, $S \subset A$ a multiplicative set ($1 \in S$, and $st \in S\ \forall s, t \in S$).

Introduce th following relation $\sim$ on $A \times S$:

$$(a, s) \sim (b, t) \iff y \in S \text{ such that } u(at - bs) = 0$$

(write a/s for the equivalence class of (a, s).

Then, the *ring of fractions of A with respect to S* is

$$S^{-1}A = (A \times S)/\sim$$

with ring op'ns defined by the usual arithmetic op'ns on fractions:

$$\frac{a}{s} \pm \frac{b}{t} = \frac{(at \pm bs)}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

Proposition 6.1. i. $\sim$ is an equivalence relation

ii. the ring op'ns are well defined, and $S^{-1}A$ is a ring

iii.

$$\begin{aligned} \psi : A &\longrightarrow S^{-1}A \\ a &\longmapsto a/1 \end{aligned}$$

is a ring homomorphism.

Example . TODO

Lemma 6.2. For $f \in A$, write $S = \{1, f, f^2, \ldots\}$, and $A_f \cong A[X]/(Xf - 1)$.

Then

$$A_f \cong \frac{A[X]}{(Xf - 1)}$$

Proof. Define the homomorphism

$$\begin{aligned} \psi : A[X] &\longrightarrow A_f \\ a &\longmapsto a/1 \\ X &\longmapsto 1/f \end{aligned}$$

By the 1st isomorphism theorem:

$$\begin{tikzcd} A[X] \arrow[r, "\psi"] \arrow[d, "\phi"'] & A_f \\ A[X]/\ker(\psi) \arrow[ur, "\eta"'] & \end{tikzcd}$$

Thus we want to prove that $ker(\psi) = (Xf - 1)$, so that $\frac{A[X]}{ker(\psi)} = \frac{A[X]}{(Xf-1)}$, and the lemma is proven.

First, observe that $\psi(Xf - 1) = \psi(X)\psi(f) - \psi(1) = \frac{1}{f}\frac{f}{1} - 1 = 1 - 1 = 0$, so $Xf - 1 \in ker(\psi)$, ie. $(Xf - 1) \subseteq ker(\psi)$.

Now, we want to prove that $ker(\psi) \subseteq (Xf - 1)$.

Take $h \in ker(\psi)$, will prove that $h \in (Xf - 1) \quad h \in ker(\psi)$, and thus $ker(\psi) \subseteq (Xf - 1)$.

Want to prove that $h(X)$ is a multiple of $(Xf - 1)$.

Let

$$h(X) = a_n X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + a_0$$

multiply $h(X)$ by f^n:

$$f^n \cdot h(X) = a_n(f^n X^n) + a_{n-1}f(f^{n-1}X^{n-1}) + a_{n-2}f^2(f^{n-2}X^{n-2})\ldots$$

Note that since $\forall\ i \geq 1,\ \ f^i X^i = (Xf - 1)\cdot(f^{i-1}X^{i-1} + f^{i-2}X^{i-2} + \ldots + 1)$, then $f^i X^i \equiv 1 \pmod{Xf - 1}$.

So,

$$f^n \cdot h(X) = \underbrace{a_n(1) + a_{n-1}f(1) + a_{n-2}f^2(1) + \ldots + a_0 f^n}_{C \quad \text{(constant)}} \pmod{Xf - 1}$$

$$\begin{aligned} &\Longrightarrow \quad f^n \cdot h(X) = C \pmod{Xf - 1} \\ &\Longleftrightarrow \quad f^n \cdot h(X) = Q(X)\cdot(Xf - 1) + C \end{aligned}$$

Want to remove C, but it is non-zero. Note that in A_f (ring of fractions), $a^n = 0$ iff $\ k$ such that $f^k \cdot a = 0$ in A.

So, multiply both sides by f^k:

$$f^k \cdot f^n \cdot h(X) = f^k \cdot (Q(X)\cdot(Xf - 1) + C)$$

$$\underbrace{f^k f^n}_{f^{nk}} \cdot h(X) = \underbrace{f^k Q(X)}_{Q'(X)} \cdot (Xf - 1) + \underbrace{f^k C}_{0}$$

$$\begin{aligned} &\Longrightarrow \quad f^{n+k} \cdot h(X) = Q'(X) \cdot (Xf - 1) \\ &\Longleftrightarrow \quad f^{n+k} \cdot h(X) \equiv 0 \pmod{Xf - 1} \end{aligned}$$

multiply it by X^{n+k}:

$$X^{n+k} \cdot f^{n+k} \cdot h(X) \equiv X^{n+k} \cdot 0 \pmod{Xf - 1}$$

$$(Xf)^{n+k} \cdot h(X) \equiv 0 \pmod{Xf - 1}$$

Now, since we had $Xf \equiv 1$ in $\frac{A[X]}{(Xf-1)}$,

$$(1)^{n+k} \cdot h(X) \equiv 0 \pmod{Xf - 1}$$

$$\Longrightarrow \quad h(X) \equiv 0 \pmod{Xf - 1}$$

By definition this is saying $h(X) \in (Xf - 1)\ \forall\ k \in ker(\psi)$.

Thus $ker(\psi) \subseteq (Xf - 1)$.

Initially we saw that $(Xf - 1) \subseteq ker(\psi)$. Therefore $ker(\psi) = (Xf - 1)$.

Hence,

$$A_f \cong \frac{A[X]}{(Xf - 1)}$$

□

Given a ring homomorphism $\psi : A \longrightarrow B$, there is a correspondence

$$e : \{\text{ideals of } A\} \longrightarrow \{\text{ideals of } B\}$$

given by $e(I) = \psi(I)B = IB$ (called *extension*),
and

$$r : \{\text{ideals of } B\} \longrightarrow \{\text{ideals of } A\}$$

given by $r(J) = \psi^{-1}J$ (called *restriction*, written $A \cap J$).
Set $B = S{-}1A$. Then $S^{-1}I = e(I) = \psi(I)B$.

Proposition 6.3. a. $\forall$ ideal J of $S^{-1}A$, $e(r(J)) = J$

b. $\forall$ ideal I of A,

$$r(e(I)) = \{a \in A \mid as \in I \text{ for some } s \in S\}$$

c. if P prime and $P \cap S = \emptyset$,
then $e(P) = S^{-1}P$ is a prime ideal of $S^{-1}A$.

Proof. □

Corollary 6.3. for an ideal I of A, the necessary and sufficient condition for $r(e(I)) = I$ is

$$as \in I \Longrightarrow a \in I \quad \forall s \in S$$

6.2 Localization

If P prime ideal, then $S = A \setminus P$ is a multiplicative set. Define the set $A_P = S^{-1}A$.

Proposition 6.4. $a/s \in A_P$ is a unit of $A_P \iff a \notin P$.

Therefore A_P is a *local ring*, with maximal ideal $e(P) = PA_P$.

The local ring (A_P, PA_P) is called the *localization* of A at P.

Proof. $\Longleftarrow$ (if $a \notin P \implies a/s \in A_P$ is a unit)

if $a \notin P$, then by definition of S, $a \in S$.

In the localization A_P, every element of S is invertible.

The inverse of $\frac{a}{s}$ is $\frac{s}{a}$
$\longrightarrow \quad \frac{a}{s} \cdot \frac{s}{a} = 1$, thus $a/s \in A_P$ is a unit.

$\Longrightarrow$ (if a/s unit $\implies a \notin P$)

Suppose a/s a unit; then $\frac{b}{t} \in A_P$ such that $\frac{a}{s} \cdot \frac{b}{t} = 1$.

By definition of equality in localization, $\frac{ab}{st} = 1$ means $u \in S$ such that

$$u(ab \cdot 1 - st \cdot 1) = 0 \implies uab = ust \qquad \text{(eq.6.4)}$$

with $u, s, t \in S$.

Since $S = A \setminus P$ and P is a prime ideal,
the products of elements outside P must also be outside of P.

$$u, s, t \notin P \implies ust \notin P$$

At eq.6.4, we know that $uab = ust \notin P$, thus $uab \notin P$.

If $uab \notin P$, then $a \notin P$.

Next we will prove that A_P is a local ring.

A ring is local if it has exactly one maimal ideal (1.13); so we show that the set of non-units forms an ideal.

a/s is not a unit iff $a \in P$. Let $m = \{a/s | a \in P,\ s \notin P\} = PA_P$. Want to show that PA_P is the unique maximal ideal:

- it's an ideal: let $\frac{a}{s}, \frac{b}{t} \in PA_P$ with $a, b \in P$.
  Then $\frac{a}{b} + \frac{b}{t} \in PA_P$ with numerator in P.
  If multiply a fraction in PA_P by any fraction in A_P, the numerator stays in P, thus PA_P is an ideal.
- every element outside of PA_P is a unit (proven at the beginning of this proposition's proof ($\Longleftarrow$)).

- maximality: every ideal strictly larger than PA_P must contain a unit. If an ideal contains a unit, it must be the entire ring A_P.
  $\Longrightarrow$ therefore, PA_P is the unique maximal ideal.

□

In other words:

Localization: formal way to 'force' certain elements to have inverses.

It's the smallest ring that contains A and makes all the elements of S invertible.

There is a natural map $\psi : A \longrightarrow S^{-1}A$, by $f(a) = \frac{a}{1},\ f(s) = \frac{1}{s}$ for $s \in S$.

6.3 Localization commutes with taking quotients

Let A ring, S multiplicative set, I ideal.

Write T for the image of S in A/I.

Then $S^{-1}I = I \cdot S^{-1}A$ is an ideal of $S^{-1}A$, can take the quotient ring $S^{-1}A/S^{-1}I$.

$\longleftrightarrow$ can take the quotient A/I and then localize to get $T^{-1}(A/I)$.

Corollary 6.7.

$$T^{-1}(A/I) \cong \frac{S^{-1}A}{S^{-1}I}$$

In particular, for P prime ideal,

$$\frac{A_P}{PA_P} = k(P) = Frac(\underbrace{A/P}_{\substack{\text{integral}\\ \text{domain}}})$$

From 6.4, A_P: local ring, PA_P: unique maximal ideal. $k(P)$ and $Frac(A/P)$ are field of fractions.

Proof. the quotient ring A/I can be viewed as an A-module and

$$\underbrace{T^{-1}(A/I)}_{\text{ring of fractions}} \quad \cong \quad \underbrace{\frac{S^{-1}A}{S^{-1}I}}_{\text{module of fractions}}$$

The **??**.i, gives an isomorphism of modules

$$T^{-1}(A/I) = S^{-1}(A/I) \cong \frac{S^{-1}A}{S^{-1}I}$$

it's easy to see that this is a ring homomorphism. □

7 Exercises

For the exercises, I follow the assignments listed at [3].

The exercises that start with **R** are the ones from the book [2], and the ones starting with **AM** are the ones from the book [1].

7.1 Exercises Chapter 1

Exercise R.1.1. Ring A and ideals I, J such that $I \cup J$ is not an ideal. What's the smallest ideal containing I and J?

Proof. Take ring $A = \mathbb{Z}$. Set $I = 2\mathbb{Z},\ J = 3\mathbb{Z}$.

$I,\ J$ are ideals of A $(= \mathbb{Z})$. And $I \cup J = 2\mathbb{Z} \cup 3\mathbb{Z}$.

Observe that for $2 \in I,\ 3 \in J \implies 2, 3 \in I \cup J$, but $2 + 3 = 5 \notin I \cup J$.

Thus $I \cup J$ is not closed under addition; thus is not an ideal.

Smallest ideal of $\mathbb{Z}$ $(= A)$ containing I and J is their sum:

$$I + J = \{a + b | a \in I, b \in J\}$$

$gcd(2, 3) = 1$, so $I + J = \mathbb{Z}$.

Therefore, smallest ideal containing I and J is the whole ring $\mathbb{Z}$. □

Exercise R.1.5. let $\psi : A \longrightarrow B$ a ring homomorphism. Prove that ψ^{-1} takes prime ideals of B to prime ideals of A.

In particular if $A \subset B$ and P a prime ideal of B, then $A \cap P$ is a prime ideal of A.

Proof. (Recall: prime ideal is if $a, b \in R$ and $a \cdot b \in P$ (with $R \neq P$), implies $a \in P$ or $b \in P$).

Let

$$\psi^{-1}(P) = \{a \in A | \psi(a) \in P\} = A \cap P$$

The claim is that $\psi^{-1}(P)$ is prime ideal of A.

i. show that $\psi^{-1}(P)$ is an ideal of A:
   $0_A \in \psi^{-1}(P)$, since $\psi(0_A) = 0_B \in P$ (since every ideal contains 0).

   If $a, b \in \psi^{-1}(P)$, then $\psi(a), \psi(b) \in P$, so

   $$\psi(a - b) = \psi(a) - \psi(b) \in P$$

   hence $a - b \in \psi^{-1}(P)$.

   If $a \in \psi^{-1}(P)$ and $r \in A$, then $\psi(ra) = \psi(r)\psi(a) \in P$, since P is an ideal. Thus $ra \in \psi^{-1}(P)$.

   $\Longrightarrow$ so ψ^{-1} is an ideal of A.

ii. show that $\psi^{-1}(P)$ is prime:
   $\psi^{-1}(P) \neq A$, since if $\psi^{-1}(P) = A$, then $1_A \in \psi^{-1}(P)$, so $\psi(1_A) = 1_B \in P$, which would mean that $P = B$, a contradiction since P is prime ideal of B.

Take $a, b \in A$ with $ab \in \psi^{-1}(P)$; then $\psi(ab) \in P$, and since ψ is a ring homomorphism, $\psi(ab) = \psi(a)\psi(b)$.

Since P prime ideal, then $\psi(a)\psi(b) \in P$ implies either $\psi(a) \in P$ or $\psi(b) \in P$. Thus $a \in \psi^{-1}(P)$ or $b \in \psi^{-1}(P)$.

Hence $\psi^{-1}(P)$ $(= A \cap P)$ is a prime ideal of A.

□

Exercise R.1.6. prove or give a counter example:

a. the intersection of two prime ideals is prime

b. the ideal $P_1 + P_2$ generated by 2 prime ideals P_1, P_2 is prime

c. if $\psi : A \longrightarrow B$ ring homomorphism, then ψ^{-1} takes maximal ideals of B to maximal ideals of A

d. the map ψ^{-1} of Proposition 1.2 takes maximal ideals of A/I to maximal ideals of A

Proof. a. let $I = 2\mathbb{Z} = (2)$, $J = 3\mathbb{Z} = (3)$ be ideals of $\mathbb{Z}$, both prime.

Then $I \cap J = (2) \cap (3) = (6)$.

The ideal (6) is not prime in $\mathbb{Z}$, since $2 \cdot 3 \in (6)$, but $2 \neq (6)$ and $3 \neq (6)$.

Thus the intersection of two primes can not be prime.

b. $P_1 = (2)$, $P_2 = (3)$, both prime.

Then,

$$P_1 + P_2 = (2) + (3) = \{a + b | a \in P_1, b \in P_2\}$$

$\longrightarrow$ in a principal ideal domain (like $\mathbb{Z}$), the sum of two principal ideals is again principal, and given by $(m) + (n) = (gcd(m, n))$.

(recall: principal= generated by a single element)

So, $P_1 + P_2 = (2) + (3) = (gcd(2, 3)) = (1) = \mathbb{Z}$.

The whole ring is not a prime ideal (by the definition of the prime ideal), so $P_1 + P_2$ is not a prime ideal.

Henceforth, the sum of two prime ideals is not necessarily prime.

c. let $A = \mathbb{Z}$, $B = \mathbb{Q}$, $\psi : A \longrightarrow B$.

Since $\mathbb{Q}$ is a field, its only maximal ideal is (0).

Then

$$\psi^{-1}((0)) = (0) \subset \mathbb{Z}$$
$$\text{ie. } \psi^{-1}(m_B) = (m_B) \subset A$$

But (0) is not maximal in $\mathbb{Z}$, because $\mathbb{Z}/(0) \cong \mathbb{Z}$ is not a field.

Thus the preimages of maximal ideals under arbitrary ring homomorphisms need not be maximal.

d. $\psi : A \longrightarrow A/I$ quotient homomorphism, $I \subseteq A$ an ideal.

Let M a maximal ideal of A/I, then $\frac{(A/I)}{M}$ is a field (Proposition 1.3).

By the isomorphism theorems,

$$\frac{(A/I)}{M} \cong \frac{A}{\psi^{-1}(M)}$$

Since $\frac{(A/I)}{M}$ is a field, the quotient $\frac{A}{\psi^{-1}(M)}$ is a field, so $\psi^{-1}(M)$ is a maximal ideal of A.

$\implies$ under ψ, preimages of maximal ideals are maximal.

□

Exercise R.1.12.a. if I, J ideals and P prime ideal, prove that

$$IJ \subset P \iff I \cap J \subset P \iff I \text{ or } J \subset P$$

Proof. assume $I \subseteq P$ (for $J \subseteq P$ will be the same, symmetric), take $x \in IJ$, then

$$x = \sum_{k=1}^{n} a_k b_k$$

with $a_k \in I,\ b_k \in J$.

Each $a_k \in I \subseteq P$. Since P an ideal,

$$\sum_{k=1}^{n} a_k b_k \in P$$

thus $x \in P$, hence $IJ \subseteq P$.

So $I \subseteq P$ or $J \subseteq P \implies IJ \subseteq P$.

Conversely,

assume P prime and $IJ \subseteq P$.

Suppose by contradiction that $I \not\subseteq P$ and $J \not\subseteq P$.

- since $I \not\subseteq P$, $\exists a \in I$ with $a \notin P$
- since $J \not\subseteq P$, $\exists b \in J$ with $b \notin P$

Since $a \in I,\ b \in J,\ \ ab \in IJ \subseteq P$, but P is prime, so $ab \in P$ implies that $a \in P$ or $b \in P$. This contradicts a, b being taken outside of P.

Thus $I \not\subseteq P$ and $J \not\subseteq P$ are false.

So both directions are proven, hence

$$IJ \subseteq P \implies I \subseteq P \text{ or } J \subseteq P$$

□

Exercise R.1.18. Use Zorn's lemma to prove that any prime ideal P contains a minimal prime ideal.

Proof. Let P prime ideal of R.

$$S = \{Q \subseteq R \mid Q \text{ a prime ideal AND } Q \subseteq P\}$$

Goal: show that S has a minimal element, the minimal ideal contained in P.

$P \subset S$, so S is nonempty.

Let $C \subseteq S$ be a chain (= totally ordered subset) with respect to inclusion.

Define

$$Q_C = \bigcap_{Q \in C} Q$$

Clearly $Q_C \subseteq P$, since each $Q \in C$ is $Q \subseteq P$.

Since C is ordered by inclusion, it is a decreasing chain of prime ideals.

Intersection of a decreasing chain of prime ideals is again a prime ideal:

- if $ab \in Q_C$, then $ab \in Q \ \forall Q \in C$

- since Q prime, $\forall Q \in C$ either $a \in Q$ or $b \in Q$

If there were some $Q_1,\ Q_2 \in C$ with $a \in Q_1$ and $b \notin Q_2$, then by total ordering, either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.

In either case: contradiction, since the smaller one would have to contain the element that was assumed to be excluded.

Thus $\forall Q \in C$ the same element a, b must lie in all Q. $\implies$ lies in the intersection of them, Q_C.

Henceforth, Q_C is a prime ideal and lies in S, and its a lower bound of C in S.

Now, S is nonempty, and every chain in S has a lower bound in S (its intersection).

Therefore, S has a minimal element P_{min}.

By construction, P_{min} is a prime ideal $P_{min} \subseteq P$, and by minimality there are no strictly smaller prime ideals inside P.

So P_{min} is a minimal prime ideal, contained in P. □

Exercise R.1.10.

Proof. □

Exercise R.1.11.

Proof. □

Exercise R.1.4.

Proof. □

7.2 Exercises Chapter 2

Exercise R.2.9. $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is a s.e.s. of A-modules. Prove that if N, L are finite over A, then M is finite over A.

Proof. Denote the generators of L and N respectively as

$$\{l_1, \dots, l_k\} \subseteq L$$
$$\{n_1, \dots, n_p\} \subseteq N$$

By s.e.s. definition,

- α is injective (one-to-one), so

$$\forall l_i \in L,\ \exists\ x_i \in M \text{ s.th. } \alpha(l_i) = x_i$$

- β is surjective (onto), so

$$\forall n_j \in N,\ \exists\ y_j \in M \text{ s.th. } \beta(y_j) = n_j$$

We will show that $\{x_1, \dots, x_k, y_1, \dots, y_p\}$ generate M, and thus M is finite: Let $m \in M$, then $\beta(m) \in N$, and

$$\beta(m) = \sum_{j=1}^{p} a_j n_j \quad \text{with } a_j \in A$$

Take $m' \in M$, with $m' = \sum a_j y_j$, then

$$\beta(m') = \sum a_j \beta(y_j) = \sum a_j n_j = \beta(m)$$

Then, since $\beta(m) = \beta(m') \implies \beta(m - m') = 0$, thus

$$(m - m') \in \mathit{ker}(\beta)$$

By *exactness* property, since $\alpha : L \longrightarrow \mathit{ker}(\beta)$, we have $\mathit{ker}(\beta) = \mathit{im}(\alpha)$. Therefore, $\exists\ l \in L$ such that $\alpha(l) = m - m'$.
Since $\{l_i\}_k$ generate L,

$$l = \sum^{k} b_i l_i$$

thus

$$m - m' = \alpha(l) = \alpha(\underbrace{\sum b_i l_i}_{l}) = \sum b_i \underbrace{\alpha(l_i)}_{x_i} = \sum b_i x_i$$

Rearrange,

$$m = m' + \sum b_i x_i = \sum_{j=1}^{p} a_j y_j + \sum_{i=1}^{k} b_i x_i \qquad \forall m \in M$$

So, L provides k generators for the kernel part of M, N provides p "lifts" for the quotient part of M; thus M is generated by $k + p$ elements.
Thus M is finitely generated over A. □

7.3 Exercises Chapter 3

Exercise R.3.2. K a field, $A \supset K$ a ring which is finite dimensional as a K-vector space. Prove that A is Noetherian and Artinian.

Proof. $dim(A) = n < \infty$, so every ideal $\mathfrak{a}$ of A is a K-subspace of A, because if $x \in \mathfrak{a}$ and $c \in K$, then $c \cdot x \in \mathfrak{a}$.

1. Noetherian:
   let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in A.

   Since each I_i is a subspace, we have

   $$dim_K(I_1) \leq dim_K(I_2) \leq \dots \leq n$$

   where at some $i = m$ we have $dim_K(I_m) = dim_K(I_{m+1})$; then since $I_m \subseteq I_{m+1}$, we have $I_m = I_{m+1}$. So A is Noetherian.

2. Artinian:
   Similarly, if $I_1 \supseteq I_2 \supseteq \dots$ a descending chain of ideals in A.

   then
   $$n \geq dim_K(I_1) \geq dim_K(I_2) \geq \dots \geq 0$$

   where at some $i = m$ we have $dim_K(I_m) = dim_K(I_{m+1})$; then since $I_m \subseteq I_{m+1}$, we have $I_m = I_{m+1}$. So A is Artinian.

□

Exercise R.3.5. Let $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ an exact sequence. Let $M_1, M_2 \subseteq M$ be submodules of M.

Prove if the following holds or not:

$$\beta(M_1) = \beta(M_2) \text{ and } \alpha^{-1}(M_1) = \alpha^{-1}(M_2) \implies M_1 = M_2$$

Proof. Counterexample showing that it does not hold:

Let K a field, $M = K \oplus K$, $L = K$, $N = K$.

Set, for $l \in L$, $(m_1, m_2) \in M$,

$$\alpha : l \longmapsto (l, 0)$$
$$\beta : (m_1, m_2) \longmapsto m_2$$

So we have
$$0 \longrightarrow K \xrightarrow{\alpha} K^2 \xrightarrow{\beta} K \longrightarrow 0$$

Then,

$$M_1 = \{(x, x) \mid x \in K\} \qquad \sim \text{(diagonal line)}$$
$$M_2 = \{(0, x) \mid x \in K\} \qquad \sim \text{(y-axis)}$$

(Geometric interpretation: M_1, M_2 are the *diagonal line* and *y-axis* respectively; and α, β capture information about the *vertical* components (x-axis, y-axis respectively), but not about the *diagonal* way a submodule is embedded in M).

Then,

$$\beta(M_1) = \{x \mid x \in K\} = K$$
$$\beta(M_2) = \{x \mid x \in K\} = K$$

thus, $\beta(M_1) = \beta(M_2)$.

For M_1, $(l, 0) \in M$ iff $l = 0$, thus $\alpha^{-1}(M_1) = \{0\}$,
for M_2, $(l, 0) \in M$ iff $l = 0$, thus $\alpha^{-1}(M_2) = \{0\}$,
thus $\alpha^{-1}(M_1) = \alpha^{-1}(M_2)$.

So we've seen that

$$\beta(M_1) = \beta(M_2)$$
$$\alpha^{-1}(M_1) = \alpha^{-1}(M_2)$$

while having $M_1 \neq M_2$. □

Exercise R.3.3. Let A a ring, $I_1, \dots, I_k$ ideals such that each A/I_i is a Noetherian ring. Prove that $\bigoplus A/I_i$ is a Noetherian A-module, and deduce that if $\bigcap I_i = 0$ then A is also Noetherian.

Proof. i. by Corollary R.3.5 (i), if M_i Noetherian modules, then $\bigoplus M_i$ is Noetherian. $\Longrightarrow$ thus $\bigoplus A/I_i$ is Noetherian.

ii. Take the canonical homomorphism

$$\phi : A \longrightarrow \bigoplus_{i=1}^{n} A/I_i$$

by $\phi(a) = (a + I_1, a + I_2, \dots, a + I_n)$.

ϕ is injective: $ker(\phi) = \{a \in A | a \in I_i \forall i\}$.

Since we're given $\cap I_i = 0$, then $ker(\phi) = \cap I_i$, and ϕ is injective.

Thus, ϕ is the isomorphism $A \cong im(\phi)$, where $im(\phi)$ is an A-submodule of $\bigoplus A/I_i$.

We know that any submodule of a Noetherian module is Noetherian, thus, since

- A/I_i is Noetherian by hypothesis of the exercise
- $A \cong im(\phi)$
- $im(\phi)$ is an A-submodule of $\bigoplus A/I_i$

then, A is Noetherian.

□

Exercise R.3.4. Prove that if A is a Noetherian ring and M a finite A-module, then there exists an exact sequence $A^q \xrightarrow{\alpha} A^p \xrightarrow{\beta} M \longrightarrow 0$. That is, M has a presentation as an A-module in terms of finitely many generators and relations.

Proof. since M fingen $\implies$ generators $\{m_1, \ldots, m_2\} \subseteq M$ span M.

Let β be a surjective A-linear map, which forms a free A-module of rank p onto M:

$$\beta : A^p \longrightarrow M$$
$$(a_1, \ldots, a_p) \longmapsto \sum_{i=1}^{p} a_i m_i$$

Let $K = ker(\beta)$. By the 1st Isomorphism Theorem,

$$M \cong A^p/K$$

Since A is a Noetherian ring, then every free A-module of finite rank (eg. A^p) is a Noetherian module.

Every submodule of a Noetherian module is fingen.

$\implies$ since $K \subseteq A^p$, $\implies$ K $(= ker(\beta))$ is fingen.

Since K fingen, let $\{k_1, \ldots, l_q\}$ be generators of K.

Define $\psi : A^q \longrightarrow K$.

Compose it with the inclusion map $i : K \longrightarrow A^p$,

$$\alpha = i \circ \psi : \; A^q \longrightarrow A^p$$

So we have the whole sequence $A^q \xrightarrow{\alpha} A^p \xrightarrow{\beta} M \longrightarrow 0$, where

- β is surjective
- $im(\alpha) = ker(\beta)$

so that it is a exact sequence, thus, M has a finite presentation. □

7.4 Exercises Chapter 4

Exercise R.4.1.a. $k[X^2] \subset k[X]$ is a finite extension, hence integral. Find the integral dependence relation for any $f \in k[X]$.

Proof. $\forall f(X) \in k[X]$ can be uniquely decomposed into its even and odd parts:

$$f(X) = p(X^2) + X \cdot q(X^2)$$

with $p(X^2),\; q(X^2) \in k[X^2]$, and
$p(X^2)$: sum of all terms with even exponents
$q(X^2)$: sum of all terms with odd exponents, and then factoring out X.

(Observation: this is used in FRI cryptographic protocol
https://github.com/arnaucube/math/blob/master/notes_fri_stir.pdf)

Rearrange it

$$\begin{aligned} f(X) - p(X^2) &= X \cdot q(X^2), \quad \text{square:} \\ (f(X) - p(X^2))^2 &= X^2 \cdot q(X^2)^2 \\ f(X)^2 - 2p(X^2)f(X) + p(X^2)^2 &= X^2 \cdot q(X^2)^2 \\ f(X)^2 \underbrace{-[2p(X^2)]}_{a_1} f(X) + \underbrace{[p(X^2)^2 - X^2 \cdot q(X^2)^2]}_{a_0} &= 0 \end{aligned}$$

Denote the last polynomial as $P(T) \in k[X^2]$, where $f(X)$ is a root of $P(T)$.

The integral dependence relation for any $f \in k[X]$ is given by the monic polynomial from R.4.1.ii, in this case $T^2 + a_1T + a_0 = 0$ with $a_i \in k[X^2]$.

We have that

$$\begin{aligned} a_1 &= -2p(X^2) \\ a_0 &= p(X^2)^2 - X^2q(X^2)^2 \end{aligned}$$

So for example, for $f(X) = X^3 + X^2 + X + 1$:

$$\begin{aligned} f(X) &= (X^2 + 1) + X(X^2 + 1) \\ (f(X) - (X^2 + 1))^2 &= X^2(X^2 + 1)^2 \\ (f(X) - p(X))^2 &= X^2(q(X))^2 \end{aligned}$$

□

Exercise R.4.5. Let $A = k[X, Y]/(Y^2 - X^2 - X^3)$. Prove that the normalization of A is $k[t]$ where $t = Y/X$.

Proof. $A = k[X, Y]/(Y^2 - X^2 - X^3)$, express X and Y in terms of t:

Since $t = Y/X$ then $Y = tX$, and combined with $Y^2 = X^2 + X^3$, then

$$\begin{aligned} (tX)^2 &= X^2 + X^3 \\ t^2X^2 &= X^2 + X^3, \quad \text{assuming } X \neq 0: \\ t^2 &= 1 + X, \text{ thus} \\ X &= t^2 - 1 \in k[X] \end{aligned}$$

Then, $Y = tX = t(t^2 - 1) = t^3 - t \in k[X]$.

Hence $X, Y \in k[X]$.

Therefore, $k[X, Y]/(Y^2 - X^2 - X^3) \subseteq k[t]$.

By R.4.6 (Noether normalization lemma), to show that $k[t]$ is the *normalization*, must show that $k[t]$ is *integral* over A.

From $X = t^2 - 1 \implies t^2 - 1 - X = 0 \implies t^2 - (1 + X) = 0$.

$(1 + X) \in A$, so t satisfies the monic polynomial

$$P(T) = T^2 - (1 + X) \in A[T]$$

Thus t is integral over A.

Since $k[t]$ is generated by t over k, and $k \subset A$, then the entire ring $k[t]$ is integral over A.

Since $k[t]$ is a polynomial ring over a field, which is a UFD, it is integrally closed (since all UFD are integrally closed).

$FracA = k(X,Y)$, since $X = t^2 - 1,\ Y = t^3 - t \implies k(X,Y) \subseteq k(t)$

and $t = Y/X \in k(X,Y)$, thus $k(X,Y) = k(t)$.

Since $k[t]$ is integrally closed and is the integral closure of A in its fraction field $k(t)$, we conclude that the normalization of A is $k[t]$. □

Exercise R.4.9. k a field, $A = k[X,Y,Z]/(X^2 - Y^3 - 1, XZ - 1)$, find $\alpha, \beta \in k$ such that A is integral over $B = k[X + \alpha Y + \beta z]$, and write a set of generators of A as a B-module.

Proof. (want to find a linear combination of the coordinates such that the original variables satisfy monic polynomials over the new ring B)

The relations defining A are

$$\begin{aligned} X^2 - Y^3 - 1 = 0 &\implies Y^3 = X^2 - 1 \quad (*) \\ XZ - 1 = 0 &\implies Z = 1/X = X^{-1} \end{aligned}$$

Thus A can be denoted as $A = k[X, Y, X^{-1}]/(Y^3 - X^2 - 1)$.

Now, Y is inegral over $k[X]$, since $Y^3 - (X^2 - 1) = 0$ is monic in Y with coefficients in $k[X]$.

Z is not integral over $k[X]$, since $Z = 1/X$ and X is not a unit in $k[X]$.

Choose $\alpha, \beta \in k$ such that X (and thus Z) becomes integral over B:
set $\alpha = 0,\ \beta = 1 \implies B = k[X + \alpha Y + \beta Z] = k[X + Z]$.

Let $w = X + Z$; since $XZ = 1$, we have

$$w = X + \frac{1}{X} \implies Xw = X^2 + 1 \implies X^2 - wX + 1 = 0 \quad (**)$$

which is monic with coefficients in $k[w]$, thus X is integral over B.

Since $Z = w - X$, Z is also integral.

Generators of A as a B-module:

we hadd $B = k[w]$ with $w = X + Z$.

From $(**)$ we have $X^2 - wX + 1 = 0$, so $X^2 = wX - 1$.

Thus any polynomial in X can be reduced to a linear form $b_1X + b_0$ with $b_i \in k[w]$. Hence it's partial basis is $\{1, X\}$.

Fitting X^2 into $(*)$,

$$\begin{aligned} &X^2 - Y^3 - 1 = 0 \\ &Y^3 = X^2 - 1 \\ &Y^3 = wX - 2 \end{aligned}$$

thus any power of Y higher than 2 can be reduced (eg. $Y^4 = Y(wX - 2) = w(XY) - 2Y$).

So its partial basis is $\{1, Y, Y^2\}$.

For Z, since $XZ = 1$ and $w = X + Z \Longrightarrow Z = w - X$, thus Z is a B-linear combination of $\{1, X\}$.

Combining the previous partial basis, the generators are

$$\{1, X\} \times \{1, Y, Y^2\} = \{1, Y, Y^2, X, XY, XY^2\}$$

□

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