Notes on FRI

arnaucube

February 2023

Abstract

Notes taken from Vincenzo Iovino [1] explanations about FRI [2], [3], [4].

These notes are for self-consumption, are not complete, don't include all the steps neither all the proofs.

An implementation of FRI can be found at https://github.com/arnaucube/fri-commitment [5].

Contents

1 Preliminaries

1.1 General degree d test

Query at points $\{x_i\}_0^{d+1}$, z (with rand $z \overset{R}{\in} \mathbb{F}$). Interpolate $p(x)$ at $\{f(x_i)\}_0^{d+1}$ to reconstruct the unique polynomial p of degree d such that $p(x_i) = f(x_i)\ \forall i = 1, \ldots, d+1$.

V checks $p(z) = f(z)$, if the check passes, then V is convinced with high probability.

This needs $d+2$ queries, is linear, $\mathcal{O}(n)$. With FRI we will have the test in $\mathcal{O}(\log d)$.

2 FRI protocol

Allows to test if a function f is a poly of degree $\leq d$ in $\mathcal{O}(\log d)$.

Note: "P *sends* $f(x)$ to V", "*sends*", in the ideal IOP model means that all the table of $f(x)$ is sent, in practice is sent a commitment to $f(x)$.

2.1 Intuition

V wants to check that two functions $g,\ h$ are both polynomials of degree $\leq d$.

Consider the following protocol:

1. V sends $\alpha \in \mathbb{F}$ to P. P sends $f(x) = g(x) + \alpha h(x)$ to V.
2. P sends $f(x) = g(x) + \alpha h(x)$ to V.
3. V queries $f(r),\ g(r),\ h(r)$ for rand $r \in \mathbb{F}$.
4. V checks $f(r) = g(r) + \alpha h(r)$. (Schwartz-Zippel lema). If holds, V can be certain that $f(x) = g(x) + \alpha h(x)$.
5. P proves that $deg(f) \leq d$.
6. If V is convinced that $deg(f) \leq d$, V believes that both g, h have $deg \leq d$.

With high probablility, α will not cancel the coeffs with $deg \geq d + 1$.

Let $g(x) = a \cdot x^{d+1}, \quad h(x) = b \cdot x^{d+1}$, and set $f(x) = g(x) + \alpha h(x)$. Imagine that P can chose α such that $ax^{d+1} + \alpha \cdot bx^{d+1} = 0$, then, in $f(x)$ the coefficients of degree $d + 1$ would cancel.

Here, P proves $g,\ h$ both have $deg \leq d$, but instead of doing $2 \cdot (d+2)$ queries ($d + 2$ for g, and $d + 2$ for h), it is done in $d + 2$ queries (for f). So we halved the number of queries.

2.2 FRI-LDT

FRI low degree testing.
Both P and V have oracle access to function f.

V wants to test if f is polynomial with $deg(f) \leq d$.

Let $f_0(x) = f(x)$.

Each polynomial $f(x)$ of degree that is a power of 2, can be written as

$$f(x) = f^L(x^2) + x f^R(x^2)$$

for some polynomials $f^L,\ f^R$ of degree $\frac{deg(f)}{2}$, each one containing the even and odd degree coefficients as follows:

$$f^L(x) = \sum_{0}^{\frac{d+1}{2}-1} c_{2i} x^i, \quad f^R(x) = \sum_{0}^{\frac{d+1}{2}-1} c_{2i+1} x^i$$

eg. for $f(x) = x^4 + x^3 + x^2 + x + 1$,

$$\left.\begin{array}{l} f^L(x) = x^2 + x + 1 \\ f^R(x) = x + 1 \end{array}\right\} \; f(x) = f^L(x^2) + x \cdot f^R(x^2)$$
$$= (x^2)^2 + (x^2) + 1 + x \cdot ((x^2) + 1)$$
$$= x^4 + x^2 + 1 + x^3 + x$$

Proof generation *(Commitment phase)* P starts from $f(x)$, and for $i = 0$ sets $f_0(x) = f(x)$.

1. $\forall\ i \in \{0, log(d)\}$, with $d = deg\ f(x)$,
   P computes $f_i^L(x),\ f_i^R(x)$ for which
   $$f_i(x) = f_i^L(x^2) + x f_i^R(x^2) \qquad \text{(eq. } A_i\text{)}$$
   holds.
2. V sends challenge $\alpha_i \in \mathbb{F}$
3. P commits to the random linear combination f_{i+1}, for
   $$f_{i+1}(x) = f_i^L(x) + \alpha_i f_i^R(x) \qquad \text{(eq. } B_i\text{)}$$
4. P sets $f_i(x) := f_{i+1}(x)$ and starts again the iteration.

Notice that at each step, $deg(f_i)$ halves.

This is done until the last step, where $f_i^L(x),\ f_i^R(x)$ are constant (degree 0 polynomials). For which P does not commit but gives their values directly to V.

(Query phase) P would receive a challenge $z \in D$ set by V (where D is the evaluation domain, $D \in \mathbb{F}$), and P would open the commitments at $\{z^{2^i}, -z^{2^i}\}$ for each step i. (Recall, ”opening” means that would provide a proof (MerkleProof) of it).

Data sent from P to V

Commitments: $\{Comm(f_i)\}_0^{log(d)}$
eg. $\{Comm(f_0),\ Comm(f_1),\ Comm(f_2),\ ...,\ Comm(f_{log(d)})\}$

Openings: $\{f_i(z^{2^i}),\ f_i(-(z^{2^i}))\}_0^{log(d)}$
for a challenge $z \in D$ set by V
eg. $f_0(z),\ f_0(-z),\ f_1(z^2),\ f_1(-z^2),\ f_2(z^4),\ f_2(-z^4),\ f_3(z^8),\ f_3(-z^8),\ \ldots$

Constant values of last iteration: $\{f_k^L,\ f_k^R\}$, for $k = log(d)$

Verification V receives:

$$\text{Commitments: } Comm(f_i),\ \forall i \in \{0, log(d)\}$$
$$\text{Openings: } \{o_i, o_i'\} = \{f_i(z^{2^i}),\ f_i(-(z^{2^i}))\},\ \forall i \in \{0, log(d)\}$$
$$\text{Constant vals: } \{f_k^L,\ f_k^R\}$$

For all $i \in \{0, log(d)\}$, V knows the openings at z^{2^i} and $-(z^{2^i})$ for $Comm(f_i(x))$, which are $o_i = f_i(z^{2^i})$ and $o_i' = f_i(-(z^{2^i}))$ respectively.

V, from (eq. A_i), knows that

$$f_i(x) = f_i^L(x^2) + x f_i^R(x^2)$$

should hold, thus

$$f_i(z) = f_i^L(z^2) + z f_i^R(z^2)$$

where $f_i(z)$ is known, but $f_i^L(z^2)$, $f_i^R(z^2)$ are unknown. But, V also knows the value for $f_i(-z)$, which can be represented as

$$f_i(-z) = f_i^L(z^2) - z f_i^R(z^2)$$

(note that when replacing x by $-z$, it loses the negative in the power, not in the linear combination).

Thus, we have the system of independent linear equations

$$f_i(z) = f_i^L(z^2) + z f_i^R(z^2)$$
$$f_i(-z) = f_i^L(z^2) - z f_i^R(z^2)$$

for which V will find the value of $f_i^L(z^{2^i})$, $f_i^R(z^{2^i})$. Equivalently it can be represented by

$$\begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix} \begin{pmatrix} f_i^L(z^2) \\ f_i^R(z^2) \end{pmatrix} = \begin{pmatrix} f_i(z) \\ f_i(-z) \end{pmatrix}$$

where V will find the values of $f_i^L(z^{2^i})$, $f_i^R(z^{2^i})$ being

$$f_i^L(z^{2^i}) = \frac{f_i(z) + f_i(-z)}{2}$$
$$f_i^R(z^{2^i}) = \frac{f_i(z) - f_i(-z)}{2z}$$

Once, V has computed $f_i^L(z^{2^i})$, $f_i^R(z^{2^i})$, can use them to compute the linear combination of

$$f_{i+1}(z^{2^i}) = f_i^L(z^{2^i}) + \alpha_i f_i^R(z^{2^i})$$

obtaining then $f_{i+1}(z^{2^i})$. This comes from (eq. B_i).

Now, V checks that the obtained $f_{i+1}(z^{2^i})$ is equal to the received opening $o_{i+1} = f_{i+1}(z^{2^i})$ from the commitment done by P. V checks also the commitment of $Comm(f_{i+1}(x))$ for the opening $o_{i+1} = f_{i+1}(z^{2^i})$.
If the checks pass, V is convinced that $f_1(x)$ was committed honestly.

Now, sets $i := i + 1$ and starts a new iteration.

For the last iteration, V checks that the obtained $f_i^L(z^{2^i}),\ f_i^R(z^{2^i})$ are equal to the constant values $\{f_k^L,\ f_k^R\}$ received from P.

It needs $log(d)$ iterations, and the number of queries (commitments + openings sent and verified) needed is $2 \cdot log(d)$.

2.3 Parameters

P commits to f_i restricted to a subfield $F_0 \subset \mathbb{F}$. Let $0 < \rho < 1$ be the *rate* of the code, such that

$$|F_0| = \rho^{-1} \cdot d$$

Thm 2.1. For $\delta \in (0, 1 - \sqrt{\rho})$, we have that if V accepts, then w.v.h.p. (with very high probability) $\Delta(f_0,\ p^d) \leq \delta$.

3 FRI as polynomial commitment scheme

This section overviews the trick from [4] to convert FRI into a polynomial commitment.

Want to check that the evaluation of $f(x)$ at r is $f(r)$, which is equivalent to proving that $\exists\ Q \in \mathbb{F}[x]$ with $deg(Q) = d - 1$, such that

$$f(x) - f(r) = Q(x) \cdot (x - r)$$

note that $f(x) - f(r)$ evaluated at r is 0, so $(x - r)|(f(x) - f(r))$, in other words $(f(x) - f(r))$ is a multiple of $(x - r)$ for a polynomial $Q(x)$.

Let us define $g(x) = \frac{f(x) - f(r)}{x - r}$.

Prover uses FRI-LDT 2.2 to commit to $g(x)$, and then prove w.v.h.p that $deg(g) \leq d - 1$ ($\iff \Delta(g,\ p^{d-1} \leq \delta)$.

Prover was already proving that $deg(f) \leq d$.

Now, the missing thing to prove is that $g(x)$ has the right shape. We can relate g to f as follows: V does the normal FRI-LDT, but in addition, at the first iteration: V has $f(z)$ and $g(z)$ openings, so can verify

$$g(z) = (f(z) - f(r)) \cdot (z - r)^{-1}$$

References

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[3] Ulrich Haböck. A summary on the fri low degree test. Cryptology ePrint Archive, Paper 2022/1216, 2022. https://eprint.iacr.org/2022/1216.

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