Seminar exercises

February 2022

Solution 1.9.

1. Let $f(a) = u$, then $g(f(a)) = g(u)$, so $g \circ f$ is a function.

2. We can see that composition of functions is associative as follows:
   we know that $[f \circ g](x) = f(g(x)), \forall x \in A$,
   so,
   $$(h \circ [g \circ f])(x) = h([g \circ f](x)) = h(g(f(x)))$$

   and
   $$([h \circ g] \circ f)(x) = [h \circ g](f(x)) = h(g(f(x)))$$
   Then, we can see that
   $$h \circ (g \circ f) = h(g(f(x))) = (h \circ g) \circ f$$

Solution 1.28.

The quotient set of the equivalence relation in Example 1.27 is

$$X/\sim = \{[(x_0, y_0)], [(x_1, y_1)], \ldots, [(x_n, y_n)]\}$$

Yes, it is isomorphic to the cosets of the *nth* roots of unity, which are $\mathbb{G}_n = \{w_k\}_{k=0}^{n-1}$, where $w_k = e^{\frac{2\pi i}{n}}$.

Solution 2.2.

To prove that the inverse x^{-1} is unique, assume x^{-1} and $\tilde{x}^{-1}$ are two inverses of x.
By the definition of the inverse, we know that $x \cdot x^{-1} = e$. And by the definition of the unit element, we know that $x \cdot e = x$.
Then,
$$x^{-1} \cdot (x \cdot \tilde{x}^{-1}) = x^{-1} \cdot e = x^{-1}$$
and
$$(x^{-1} \cdot x) \cdot \tilde{x}^{-1} = e \cdot \tilde{x}^{-1} = \tilde{x}^{-1}$$
By associativity property of groups, we know that

$$x^{-1} \cdot (x \cdot \tilde{x}^{-1}) = (x^{-1} \cdot x) \cdot \tilde{x}^{-1}$$

so,

$$x^{-1} \cdot e = e \cdot \tilde{x}^{-1}$$

which is

$$x^{-1} = \tilde{x}^{-1}$$

So, for any $x \in G$, the inverse x^{-1} is unique.

Solution 2.5.

Let $\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$, $\beta = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$, then,

$$\alpha \cdot \beta = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$$

and

$$\beta \cdot \alpha = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$$

So, we can see that

$$\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$$

so, $\alpha \cdot \beta \neq \beta \cdot \alpha$.

Solution 2.26.

We want to prove that $f : G \to H$ is a *monomorphism* iff $\ker f = \{e\}$.
We know that f is a *monomorphism* (*injective*) iff $\forall a, b \in G$, $f(a) = f(b) \Rightarrow a = b$.
Let $a, b \in G$ such that $f(a) = f(b)$. Then

$$f(a)f(b)^{-1} = f(b)(f(b))^{-1} = e$$

$$f(a)f(b^{-1}) = e$$

$$f(ab^{-1}) = e$$

as $\ker f = \{e\}$, then we see that $ab^{-1} = e$, so $a = b$. Thus f is a *monomorphism.*