Notes on "A book of Abstract Algebra", Charles C. Pinter

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Abstract

Notes on *"A book of Abstract Algebra - by Charles C. Pinter"*, is a *LaTeX* version of handmade notes taken while reading the book. It contains only some definitions and theorems (without proofs), so it is highly recommended to read the actual book instead of the current notes. Additionally, some theorems and concepts are extended with notes from other resources from outside the book.
This is an unfinished and 'work in progress' document.

Contents

1 Groups

Def 1.1 (Group)**.** A set G with an operation $*$ which satisfies the axioms:

i. $*$ is *associative*

ii. *(identity element)* there is an element $e \in G$ s.t. $a * e = a$ and $e * a = a$ $\forall a \in G$

iii. *(inverse)* $\forall a \in G$, there is an element $a^{-1} \in G$ s.t. $a*a^{-1} = e$ and $a^{-1}*a = e$

Def 1.2 (Abelian Group)**.** A group G is said to be *commutative* if $\forall a, b \in G$, $ab = ba$. A commutative group is also called *Abelian*.

Def 1.3 (Order of an element)**.** In a group G, the order of an element $a \in G$ is the least positive integer n such that $a \cdot a \cdots a = a^n = e$. It is represented by $ord(a)$.

Def 1.4 (Order of a group)**.** Order of a group G, is the number of elements in G. It is represented by $|G|$.

Def 1.5 (Cyclic group)**.** Let G be a group, and $a \in G$. If G consists of all the powers of a and nothing else:

$$G = \{a^n : n \in \mathbb{Z}\}$$

then, G is called a *cyclic group*, and a is called its *generator*.
The group G generated by a is defined by $G = \langle a \rangle$.

Thm 1.6. The *order of a cyclic group* is the same as the *order of it's generator*. In other words, for a cyclic group, $|\langle a \rangle| = ord(a)$.

> $\langle a \rangle$ defines a cyclic group generated by a. $\langle a \rangle = \{e, a, a^2, ..., a^{n-1}\}$
>
> $|\langle a \rangle|$ defines the order of the cyclic group generated by a.

Thm 1.7. Every subgroup of a cyclic group is cyclic.

2 Subgroups

Def 2.1 (Subgroup)**.** Let G be a group, and H a non-empty subset of G. If

i. the idenity e of G is in H.

ii. H is closed with respect to the operation. Which is for $a, b \in H$, $ab \in H$.

iii. H is closed with respect to inverses. Which is for $a \in H$, $a^{-1} \in H$.

we call H a *subgroup* of G. The operation of H is the same as the operation of G.

Thm 2.2. Every subgroup of a cyclic group is cyclic.

3 Functions

Def 3.1 (Function)**.** If A and B are sets, then a function from A to B is a rule which to every element x in A assigns a unique element y in B.
Functions are represented by $f : A \to B$, where $\forall a \in A \Rightarrow f(a) \in B$.

Def 3.2 (Injective (monomorphism))**.** A function $f : A \to B$ is called *injective* if each element of B is the image of no more than one element of A.

Def 3.3 (Surjective (epimorphism))**.** A function $f : A \to B$ is called *surjective* if each element of B is the image of at least one element of A.

Def 3.4 (Bijective (isomorphism))**.** A function $f : A \to B$ is called *bijective* if it is both *injective* and *surjective*.
A function $f : A \to B$ has an inverse iff it is *bijective*. In that case, the inverse f^{-1} is a bijective function from B to A.

Def 3.5 (Composite function)**.** A function $f : A \to B$ and $g : B \to C$ be functions. The *composite function* denoted by $g \circ f$ is a function from A to C defined as follows:

$$[g \circ f](x) = g(f(x)), \forall x \in A$$

Def 3.6 (Permutation)**.** By a *permutation* of a set A we mean a *bijective function from A to A*, that is, a one-to-one correspondence between A and itself.
The set of all the permutations of A, with the operation $\circ$ of composition, is a group.
For any positive integer n, the symmetric group on the set $1, 2, 3, ..., n$ is called the *symmetric group on n elements*, and is denoted by S_n.

4 Isomorphism

Def 4.1 (Isomorphism)**.** Let G_1 and G_2 be groups. A bijective function $f : G_1 \to G_2$ with the property that for any two elements $a, b \in G_1$,

$$f(ab) = f(a)f(b)$$

is called an *isomorphism* from G_1 to G_2.
If there exists an isomorphism from G_1 to G_2, we say that G_1 is *isomorphic* to G_2, symbolized by $G_1 \cong G_2$.

Thm 4.2 (Cayley's Theorem)**.** Every group is isomorphic to a group of permutations.

Thm 4.3. (Isomorphism of cyclic groups)

i. For every positive integer n, every cyclic group of order n is isomorphic to $\mathbb{Z}_n$. Thus, any two cyclic groups of order n are isomorphic to each other.

ii. Every cyclic group of order infinity is isomorphic to $\mathbb{Z}$, and therefore any two cyclic groups of order infinity are isomorphic to each other.

5 Cosets

Def 5.1 (Coset)**.** Let G be a group, and H a subgroup of G. For any element a in G, the symbol aH denotes the set of all products ah, as a remains fixed and h ranges over H. aH is caled a *left coset* of H in G.
In similar fashion, Ha denotes the set of all products ha, as a remains fixed an h ranges over H. Ha is called a *right coset* of H in G.

Thm 5.2. If Ha is any coset of H, there is a one-to-one correspondence from H to Ha (there is a bijection between H and Ha).
If $a \in G$, then $|H| = |Ha|$.

Thm 5.3 (Lagrange's theorem)**.** Let G be a finite group, and H any subgroup of G. The order of G is a multiple of the order of H.

Lagrange's theorem can be easily seen by the facts that:

i. cosets partition the group G

ii. $|Ha| = |H|$ (each coset has the same order as H).

By consequence,

Thm 5.4. If G is a group with a prime number p of elements, then G is a cyclic group. Furthermore, any element $a \neq e$ in G is a generator of G.

Thus,

Thm 5.5. The order of any element of a finite group divides the order of the group.

Def 5.6 (Index of H in G)**.** Number of cosets of H in G. Represented by $(G : H)$. Combined with *Lagrange Theorem*, we know that $|G| = |H| \cdot |G : H|$, so,

$$(G : H) = \frac{|G|}{|H|}$$

6 Homomorphisms

Def 6.1 (Homomorhism)**.** If G and G are groups, a *homomorphism* from G to H is a function $f : G \to H$ s.t. for any two elements $a, b \in G$,

$$f(ab) = f(a)f(b)$$

If there exists a homomorphism from G *onto* H, we say that H is a *homomorphic image* of G.

Note: an *isomorphism* is a *bijective homomorphism*.
Example of an *homomorphism*: $f : \mathbb{Z}_6 \to \mathbb{Z}_3$.

Thm 6.2. Let G and G be groups, and $f : G \to H$ a homomorphism. Then

i. $f(e) = e$

ii. $f(a^{-1}) = [f(a)]^{-1}, \quad \forall a \in G$

Def 6.3 (Conjugate)**.** A *conjugate* of a is any element of the form xax^{-1}, where $x \in G$.

Def 6.4 (Normal subgroup)**.** Let H be a subgroup of a group G. H is called a *normal* subgroup of G if it is closed with respect to conjugates, that is, if
for any $a \in H$ and $x \in G$, $xax^{-1} \in H$.
Alternatively, we can see that H is a *normal* subgroup iff $\forall a \in G, aH = Ha$.
In an abelian group, every subgroup is normal.

Def 6.5 (Kernel)**.** Let $f : G \to H$ be a homomorphism. The *kernel* of f is the set K of all the elements of G which are carried by f onto the neutral element of H. That is,

$$K = x \in G : f(x) = e$$

For every homomorphism, the $e \in G$ maps to $e \in H$, so the *kernel* is never empty, it always contains the identity e_G, and if the kernel only contains the identity, then f is one-to-one (injective).

7 Quotient Groups

Quotient group construction is useful as a way of actually manufacturing all the homomorphic images of any group G. Additionally, we can often choose H so as to "factor out" unwanted properties of G, and prserve in G/H only "desirable" traits.

Def 7.1 (Coset multiplication)**.** The coset of a, multiplied by the coset of b, is defined to be the coset of ab. In symbols, $Ha \cdot Hb = H(ab)$.

Thm 7.2. Let H be a normal subgroup of G. If $Ha = Hc$ and $Hb = Hd$, then $H(ab) = H(cd)$.

Def 7.3. G/H denotes the set which consists of *all the cosets of* H.
Thus, if $Ha, Hb, Hc, \ldots$ are cosets of H, then $G/H = \{Ha, Hb, Hc, ...\}$.

Thm 7.4 (Quotient group)**.** G/H with coset multiplication is a group.

Thm 7.5. G/H is a homomorphic image of G.
Conversely, every homomorphic image of G is a quotient group of G.

Thm 7.6. Let G be a group and H a subgroup of G. Then

i. $Ha = Hb$ iff $ab^{-1} \in H$

ii. $Ha = H$ iff $a \in H$

8 Rings

Def 8.1 (Ring)**.** A set A with operations called *addition* and *multiplication* which satisfy the following axions:

i. A with addition alone is an abelian group.

ii. Multiplication is associative.

iii. Multiplication is distributive over addition. That is, $\forall a, b, c \in A$,

$$a(b + c) = ab + ac$$

$$(b + c)a = ba + ca$$

Def 8.2 (Commutative ring)**.** By definition, addition is commutative in every ring but multiplication is not. When multiplication also is commutative in a ring, we call that ring a *commutative* ring.

Def 8.3 (Unity)**.** A ring does not necessarily have a neutral element for multiplication. If there is in A a neutral element for mulitplication, it is called the *unity* of A, and is denoted by the symbol 1.
If A has a unity, we call A a *ring with unity.*

Def 8.4 (Field)**.** If A is a commutative ring with unity in which every nonzero element is invertible, A is called a *field.*

Def 8.5 (Divisor of zero)**.** In any ring, a nonzero element a is called a *divisor of zero* if there is a nonzero element b in the ring such that the product ab or ba is equal to zero.

Def 8.6 (Cancellation property)**.** A ring is said to have the cancellation property if $ab = ac$ or $ba = ca$ implies $b = c$ for any elements a, b, and c in the ring if $a \neq 0$.

Thm 8.7. A ring has the *cancellation property* iff it has no *divisors of zero.*

Def 8.8 (Integral domain)**.** An *integral domain* is defined to be a commutative ring with unity having the cancellation property.

Every field is an integral domain, but the converse is not true (eg. $\mathbb{Z}$ is an integral domain but not a field).

Def 8.9 (Ideal)**.** A nonempty subset B of a ring A is called an *ideal* of A if B is closed with respect to addition and negatives, and B absorbs products in A.

WIP: covered until chapter 18, work in progress.